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Strong-Cut Enumerative Procedure for Extreme Point Mathematical Programming Problems

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Summary: In this paper a "Strong-Cut Enumerative Procedure" for solving Extreme Point Mathematical Programming Problem:

$$\text{Max } CX$$

subject to $AX = b$
and that X is an extreme point of

$$DX = d, \quad X \geq 0,$$

is developed. The procedure will avoid the investigation of many of the extreme points of $DX = d$, $X \geq 0$ and also alternative optimas of different best extreme points of $DX = d$, $X \geq 0$ will not be needed. The algorithm is expected to work very efficiently.

Zusammenfassung: In dieser Arbeit "Strong-Cut Enumerative Procedure for Extrem Point Mathematical Programming Problem" wird ein sehr effizientes enumeratives Verfahren zur Lösung des Problems

$$\text{Max}\{cx \mid Ax = b\,; \quad x \text{ Extremzahl von } Dx = d, \quad x \geq 0$$

entwickelt. Extrempunkte von $Dx = d$, $x \geq 0$ werden in systematischer Weise abgesucht, bis Zulässigkeit in $Ax = b$ erreicht ist. Die dabei benutzten Kriterien vermeiden die Untersuchung vieler nichtoptimaler Extrempunkte und die Bestimmung alternativer Optimalpunkte von $Dx = d$, $x \geq 0$.

Introduction

The general form of Extreme Point Mathematical Programming Problem is:

$$\left.\begin{array}{l} \text{Max } CX \\ \text{subject to } AX = b \\ \text{and } X \text{ is an extreme point of} \\ DX = d,\ X \geq 0 \end{array}\right\} \quad \text{Problem (I)}$$

where A is $m \times n$, X is $n \times 1$, b is $m \times 1$, C is $1 \times n$, D is $p \times n$, d is $p \times 1$, and 0 is $n \times 1$.

In [*Kirby-Love-Swarup*, 1972] a cutting plane algorithm is developed in which one moves from extreme point to extreme point of the problem:

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$$\left.\begin{array}{l} \text{Max } CX \\ \text{subject to } FX = f \\ \qquad\qquad X \geq 0 \\ \text{where } F = \begin{bmatrix} A \\ D \end{bmatrix} \text{ and } f = \begin{bmatrix} b \\ d \end{bmatrix} \end{array}\right\} \qquad \text{Problem (II.1)}$$

till extreme point of $DX = d$, $X \geq 0$ is reached. In this procedure, difficulties of testing whether any optimal extreme point solution of (II.1) is an extreme point of $DX = d$, $X \geq 0$ or not and of finding alternative optimas at every stage were faced.

In [*Kirby-Love-Swarup*, 1971] another cutting plane algorithm was developed wherein difficulty of finding whether an optimal extreme point solution of (II.1) is an extreme point of $DX = d$, $X \geq 0$ or not was got rid off. In this paper authors moved from extreme point to extreme point of the problem:

$$\left.\begin{array}{l} \text{Max } CX \\ \text{subject to } DX = d \\ \qquad\qquad X \geq 0 \end{array}\right\} \qquad \text{Problem (II.2)}$$

till feasibility in $AX = b$ is achieved.

In [*Kirby-Love-Swarup*, 1970] extreme points of (II.2) are enumerated till feasibility in $AX = b$ is satisfied.

Here in this paper, calculations of many of the extreme points of $DX = d$, $X \geq 0$ will be avoided by introducing a 'strong cut'. And by following the enumerative technique of [*Kirby-Love-Swarup*, 1970] alternative optimas at different stages will not be needed. This paper, therefore, will make improvement over [*Puri-Swarup*] in which the number of extreme points of $DX = d$, $X \geq 0$ which were avoided was much less. This improvement by using a 'strong cut' will be at the cost of calculating first best and second best extreme point solutions of the problem (II.1).

Theoretical Development

Problem (I) is always bounded as its any solution is to be an extreme point of $DX = d$, $X \geq 0$ and number of these extreme points is finite. To solve (I) we shall deal with (II.1) and (II.2). Problems (II.1) and (II.2) can be bounded or unbounded. When any one of them is unbounded, an inclusion of the constraint $CX \leq M$, M being sufficiently large positive number, will make the problem bounded without loosing any of the extreme points of the original problem.

Notations:

$S_1 = [X/X \text{ is an extreme point of } FX = f,\ X \geq 0]$

$S_2 = [X/X \text{ is an extreme point of } DX = d,\ X \geq 0]$

$S = [X \in S_2/AX = b]$

$X^1_{(1)}$ = Set of optimal extreme point solutions of (II.1)
$= [X^1_{11}, X^1_{12}, \ldots, X^1_{1k_1}]$
$X^1_{(2)}$ = Set of Second best extreme point solutions of (II.1)
$= [X^1_{21}, X^1_{22}, \ldots, X^1_{2k_2}]$ (See Appendix)
$u^1_{(1)}$ = Value of objective function of (II.1) at elements of $X^1_{(1)}$
$= CX^1_{11}$
$u^1_{(2)}$ = Value of objective function of (II.1) at elements of $X^1_{(2)}$
$= CX^1_{21}$
$X^2_{(1)}$ = Set of optimal extreme points of (II.2)
$= [X^2_{11}, X^2_{12}, \ldots, X^2_{1s_1}]$
$u^2_{(1)}$ = Value of objective function of (II.2) at element of $X^2_{(1)}$
$= CX^2_{11}$

Note: Definition of ith best extreme point and in particular of second best extreme point is given in Appendix.

Theorem:

Extreme points of $DX = d$, $X \geq 0$ which satisfy feasibility with respect to $AX = b$ are also extreme points of $FX = f$, $X \geq 0$ but an extreme point of $FX = f$, $X \geq 0$ need not necessarily be an extreme point of $DX = d$, $X \geq 0$. That is, $S \subseteq S_1$ [*Kirby-Lve-Swarup*, 1972; *Puri-Swarup*, 1973].

Observations:

(i) If (II.1) has no solution, then (I) will have no solution.
(ii) An extreme point X of (II.1) will be an extreme point of $DX = d$, $X \geq 0$ iff the number of non-null columns of D corresponding to non-zero basic variables in X is less than or equal to p and are linearly independent [*Kirby-Love-Swarup*, 1972].

Procedure:

Find $X^1_{(1)}$ assuming, of course, that $S_1 \neq \phi$. If $X^1_{(1)} \cap S \neq \phi$, then every $X \in X^1_{(1)} \cap S$ will be an optimal solution for (I). $X^1_{(1)} \cap S$ will be non-null iff there exists at least one element of $X^1_{(1)}$ which is also an extreme point of $DX = d$, $X \geq 0$. If $X^1_{(1)} \cap S = \phi$, we proceed to find $X^1_{(2)}$ and $u^1_{(2)}$ (see Appendix). If $X^1_{(2)} = \phi$, then (I) will have no solution. If $X^1_{(2)} \neq \phi$ and $X^1_{(2)} \cap S \neq \phi$, then every $X \in X^1_{(2)} \cap S$ will be an optimal extreme point solution of (I). If $X^1_{(2)} \neq \phi$ but $X^1_{(2)} \cap S = \phi$, we start dealing with (II.2).

Apply simplex method to (II.2) to find $X^2_{(1)}$ and $u^2_{(1)}$. As feasible region of (II.2) contains the feasible region of (II.1) and as $X^1_{(1)} \cap S = \phi$ (i.e. none of the elements of $X^1_{(1)}$ is an extreme point of $DX = d$, $X \geq 0$), it follows that $X^2_{(1)} \cap S = \phi$. Now find values $R_i (\geq u^1_{(2)})$ of the objective function of (II.2) at all adjacent extreme points of all the elements of $X^2_{(1)}$. Out of these pick up that value (say W_1) which is nearest to $u^1_{(2)}$ (i.e., $W_1 = \operatorname{Min}_i [R_i]$). Find the extreme points belonging to S_2 which correspond to W_1. Now again find values ($< W_1$ but $\geq u^1_{(2)}$) of the objective function at all the adjacent extreme points of all the elements of the set of extreme points corresponding to W_1. Out of these select that value which is nearest to $u^1_{(2)}$. Let this value be $W_2 (u^1_{(2)} \leq W_2 < W_1)$. Again determine the values ($< W_2$ but $\geq u^1_{(2)}$) of the objective function at all the adjacent points of the set of extreme points corresponding to W_2 and choose that value which is nearest to $u^1_{(2)}$. This process is repeated till we reach as close to $u^1_{(2)}$ as possible never going below $u^1_{(2)}$. Suppose W_j is the value thus reached which is nearest to $u^1_{(2)}$. Clearly $W_j \geq u^1_{(2)}$ and $W_j < W_{j-1} < W_{j-2} < \cdots < W_2 < W_1$. At this stage a cut $CX \leq W_j$ is introduced in the problem (II.2) and we get the new problem as:

$$\left.\begin{array}{l} \text{Max } CX \\ \text{subject to } DX = d \\ \qquad\qquad CX \leq W_j \\ \qquad\qquad X \geq 0 \end{array}\right\} \quad \text{Problem (II.3)}$$

As this cut viz: $CX \leq W_j$ will avoid the study of many of the unwanted extreme points of $DX = d$, $X \geq 0$, it is named as a 'strong cut'.

Find the set $X^3_{(1)} = [X^3_{11}, X^3_{12}, \ldots, X^3_{1l_1}]$ of optimal extreme point solutions of (II.3). Clearly $CX^3_{11} = W_j$, $X^3_{(1)} \neq \phi$, $X^3_{(1)} \cap S = \phi$ (i.e., elements of $X^3_{(1)}$ are not feasible with respect to $AX = b$). Let $C^3_{(1)}$ be the set of all bases of elements of $X^3_{(1)}$. As $X^3_{(1)} \neq \phi$, $C^3_{(1)} \neq \phi$. Determine the set $E^3_{(1)}$ of the basis which are adjacent to elements of $C^3_{(1)}$ and have value of the objective function less than W_j. Those elements of $E^3_{(1)}$ which yield greatest value (say $u^3_{(2)}$) of the objective function, generate the set $X^3_{(2)} = [X^3_{21}, X^3_{22}, \ldots, X^3_{2l_2}]$ of second best extreme point solutions of (II.3). $X^3_{(2)} \subseteq S_2$. If $X^3_{(2)} = \phi$, (I) has no solution. If $X^3_{(2)} \neq \phi$ and $X^3_{(2)} \cap S \neq \phi$ (i.e., there exist at least one element of $X^3_{(2)}$ which satisfies feasibility in $AX = b$), then every $X \in X^3_{(2)} \cap S$ will be optimal solution of (I). If $X^3_{(2)} \neq \phi$ but $X^3_{(2)} \cap S = \phi$ (i.e. none of the elements of $X^3_{(2)}$ satisfies feasibility in $AX = b$), then we find the set $C^3_{(2)}$ of all the bases of elements of $X^3_{(2)}$. $C^3_{(2)} \subseteq E^3_{(1)}$. Find the set $E^3_{(2)}$ of all the bases which are adjacent to the elements of $C^3_{(2)}$ and give value of the objective function less than $u^3_{(2)}$.

$$\text{Let } \begin{array}{l} H^3_{(1)} = E^3_{(1)} \\ H^3_{(2)} = E^3_{(1)} \cup E^3_{(2)} \backslash C^3_{(2)}. \end{array}$$

Determine those elements of $H^3_{(2)}$ which yield greatest value (say $u^3_{(3)}$) of the objective function. These elements will generate the set $X^3_{(3)} = [X^3_{31}, X^3_{32}, \ldots, X^3_{3l_3}]$ of third best extreme point solutions of (II.3). If $X^3_{(3)} = \phi$, (I) has no solution. If $X^3_{(3)} \neq \phi$ and $X^3_{(3)} \cap S \neq \phi$, then any $X \in X^3_{(3)} \cap S$ will be optimal solution of (I). If $X^3_{(3)} \neq \phi$ but $X^3_{(3)} \cap S = \phi$, the process is repeated with $C^3_{(i+1)}$, $E^3_{(i+1)}$ and $H^3_{(i+1)}$ in place of $C^3_{(i)}$, $E^3_{(i)}$ and $H^3_{(i)}$ starting with $i = 3$ until feasibility in $AX = b$ is achieved or some indication of no solution is obtained, where $C^3_{(i+1)}$ is the set of all the bases of the elements of $X^3_{(i+1)}$ which are $(i + 1)$th best extreme point solutions of (II.3), $E^3_{(i+1)}$ is the set of bases adjacent to elements of $C^3_{(i+1)}$ and having value of the objective function less than $u^3_{(i+1)}$ and $H^3_{(i+1)} = \bigcup_{j=0}^{i} E^3_{(j+1)} \setminus \bigcup_{j=1}^{i} C^3_{(j+1)}$. Indication of no solution at some stage (say Nth) will be given by the fact that at that stage $H^3_N = \phi$.

The process will converge in a finite number of steps since

(i) we move from one extreme point to another extreme point of (II.2) and these extreme points are finite in number.

(ii) as $u^3(i+1) < u^3_{(i)}$, no $X^3_{(i)}$ ($\subseteq S_2$ for $i \geq 2$) is repeated and

(iii) the sets $X^3_{(i)}$, $i = 1, 2, \ldots$ are disjoint.

If the 'strong cut' passes through the jth best extreme point of $DX = d$, $X \geq 0$, then clearly we save ourselves from the trouble of determining 2nd best, 3rd best, $\ldots j-1$th best extreme points of $DX = d, X \geq 0$ and jump direct to jth best extreme point.

Appendix

i) *Second Best Extreme Points:*

Out of the set of extreme points (yielding value of the objective function less than the optimal value) which are adjacent to the elements of the set of optimal extreme points, the extreme points which yield greatest value of the objective function constitute the set of second best extreme points.

Or

Second best extreme points of (II,2) are optimal extreme points of the problem:

$$\text{Max } CX$$
$$(x \in S_2 \setminus X^2_{(1)}) .$$

That is, $X^2_{(2)} \in S_2 \setminus X^2_{(1)}$

such that $u^2_{(2)} \geq CX$ for all $X \in S_2 \setminus X^2_{(1)}$

where $u^2_{(2)}$ is the value of the objective function at elements of $X^2_{(2)}$.

(ii) ith best extreme point solutions of (II.2) is the set of optimal extreme point solutions of the problem:

$$\text{Max } \; CX$$
$$\left(X \in S_2 \Big\backslash \bigcup_{j=1}^{i-1} X^2_{(j)} \right)$$

That is, $X^2_{(i)} \in S_2 \Big\backslash \bigcup_{j=1}^{i-1} X^2_{(j)}$

such that $u^2_{(i)} \geq CX$ for all $X \in S_2 \Big\backslash \bigcup_{j=1}^{i-1} X^2_{(j)}$

where $u^2_{(i)}$ is the value of the objective function at the elements of the set $X^2_{(i)}$ of ith best extreme point solutions of (II.2).

Example:

$$\text{Max } Z = x_1 + 8x_2$$

subject to

$$\begin{aligned} -7x_1 + \ \ 2x_2 &\le 4 \\ 9x_1 + 10x_2 &\le 64 \end{aligned}$$

and (x_1, x_2) is an extreme point of

$$\begin{aligned} -x_1 + 2x_2 &\le 10 \\ x_1 + 2x_2 &\le 14 \\ 2x_1 + \ \ x_2 &\le 16 \\ x_1 - \ \ x_2 &\le \ 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution:

After adding slack variables $x_3, x_4, \ldots, x_8$ we get new problem as:

$$\left.\begin{array}{l} \text{Max } Z = x_1 + 8x_2 \\ \text{subject to } \left.\begin{array}{r} -7x_1 + \ \ 2x_2 + x_3 = \ \ 4 \\ 9x_1 + 10x_2 + x_4 = 64 \end{array}\right] \equiv \left[\underset{2\times 8}{A}X = \underset{2\times 1}{b}\right. \\ \text{and } (x_1, x_2, \ldots, x_8) \text{ is an extreme point of} \\ \begin{array}{r} -x_1 + 2x_2 + x_5 = 10 \\ x_1 + 2x_2 + x_6 = 14 \\ 2x_1 + \ \ x_2 + x_7 = 16 \\ x_1 - \ \ x_2 + x_8 = \ \ 5 \end{array} \equiv \left[\underset{4\times 8}{D}X = \underset{4\times 1}{d}\right. \\ x_1, x_2, \ldots, x_8 \ge 0 \end{array}\right\} \text{Problem (I)}$$

Problem (II.1) of theory is:

$$\left.\begin{array}{l} \text{Max } Z = x_1 + 8x_2 \\ \text{subject to } \begin{array}{r} -7x_1 + \ \ 2x_2 + x_3 = \ \ 4 \\ 9x_1 + 10x_2 + x_4 = 64 \\ -x_1 + \ \ 2x_2 + x_5 = 10 \\ x_1 + \ \ 2x_2 + x_6 = 14 \\ 2x_1 + \ \ \ \ x_2 + x_7 = 16 \\ x_1 - \ \ \ \ x_2 + x_8 = \ \ 5 \\ x_1, x_2, \ldots, x_8 \ge 0 \end{array} \end{array}\right. \qquad \text{Problem (II.1)}$$

Step 1:

$$X^1_{(1)} = [X^1_{11} = (1, \tfrac{11}{2}, 0, 0, 0, 2, \tfrac{17}{2}, \tfrac{19}{2})] .$$

Number of non-null columns of D corresponding to non-zero basic variables in X^1_{11} is $= 5 > p$ ($\because$ here $p = 4$)

$$\therefore X^1_{(1)} \cap S = \phi$$
$$u^1_{(1)} = 45$$

Step 2:

$$X^1_{(2)} = [X^1_{21} = (0,2,0,44,6,10,14,7)].$$

Number of non-null columns of D corresponding to non-zero basic variables in X^1_{21} is $= 5 > p$

$$\therefore X^1_{(2)} \cap S = \phi$$
$$u^1_{(2)} = 16 .$$

Now to proceed further we consider problem (II.2) of theory which in this case is:

$$\left.\begin{aligned} \text{Max } Z &= x_1 + 8x_2 \\ \text{subject to } -x_1 + 2x_2 + x_5 &= 10 \\ x_1 + 2x_2 + x_6 &= 14 \\ 2x_1 + x_2 + x_7 &= 16 \\ x_1 - x_2 + x_8 &= 5 \\ x_1, x_2, \ldots, x_8 &\geq 0 \end{aligned}\right\} \quad \text{Problem (II.2)}$$

Step 3:

$X^2_{(1)} = [X^2_{11} = (2,6,0,0,0,0,6,9)]$, $u^2_{(1)} = 50$

Step 4:

Values R_i $(\geq u^1_{(2)})$ of the objective function of (II.2) at adjacent extreme points of $X^2_{(1)}$ are:

R_1, when d_6 enters and d_1 leaves $= 40$
R_2, when d_5 enters and d_7 leaves $= 38$
where $d_1, d_2, \ldots, d_8$ are columns of D
$W_1 = \underset{i}{\text{Min}}\,[R_i] = 38$

Extreme point of $DX = d$, $X \geq 0$ corresponding to W_1 is $(6,4,0,0,8,0,0,3)$.

Value $(< W_1$, but $\geq u^1_{(2)})$ of the objective function at extreme point adjacent to $(6,4,0,0,8,0,0,3)$ is 23.

Extreme point of $DX = d$, $X \geq 0$ yielding value 23 is $(7,2,0,0,13,3,0,0)$. It is obtained from the basis of the extreme point yielding value W_1 by entering d_6 and departing d_8.

$$W_2 = 23 > u^1_{(2)} .$$

Value $(< W_2)$ of the objective function at extreme point of $DX = d$, $X \geq 0$ adjacent to $(7,2,0,0,13,3,0,0)$ is 5, which is less than $u^1_{(2)}$.

So we will go upto W_2 only.

$$\therefore W_j = W_2 = 23 .$$

At this stage we introduce a cut $x_1 + 8x_2 \leq 23$ to the problem (II.2) and obtain the problem (II.3) of theory which is:

$$\left.\begin{array}{l}\text{Max } Z = x_1 + 8x_2 \\ \text{subject to} \left.\begin{array}{r} -x_1 + 2x_2 + x_5 = 10 \\ x_1 + 2x_2 + x_6 = 14 \\ 2x_1 + x_2 + x_7 = 16 \\ x_1 - x_2 + x_8 = 5 \\ x_1 + 8x_2 + x_9 = 23 \end{array}\right] \equiv \left[\underset{5\times 9}{D'} X = d^1\right. \\ x_1, x_2, \ldots, x_9 \geq 0 \end{array}\right\} \text{Problem (II.3)}$$

Step 5:

$$X^3_{(1)} = \begin{bmatrix} X^3_{11} = (7,2,0,0,13,3,0,0,0) \\ X^3_{12} = (0,\frac{23}{8},0,0,\frac{17}{4},\frac{33}{4},\frac{105}{4},\frac{63}{5},0) \end{bmatrix}$$

$$C^3_{(1)} = \begin{bmatrix} C^3_{11} = (d'_5, d'_6, d'_7, d'_1, d'_2), \\ C^3_{12} = (d'_5, d'_6, d'_7, d'_8, d'_2) \end{bmatrix}$$

where $d'_1, d'_2, \ldots, d'_9$ are columns of D^1.

$$E^3_{(1)} = [E^3_{11} = (d'_5, d'_6, d'_7, d'_1, d'_9),\ E^3_{12} = (d'_5, d'_6, d'_7, d'_8, d'_9)]\ .$$

Value of the objective function yielded by $E^3_{11} = 5$.
Value of the objective function yielded by $E^3_{12} = 0$.

$$\therefore u^3_{(2)} = 5\ .$$

Basis for second best extreme point of (II.3) is E^3_{11}.
Extreme point of (II.3) corresponding to E^3_{11} is (5,0,0,0,15,9,6,0,18)

$$\therefore X^3_{(2)} = [X^3_{21} = (5,0,0,0,15,9,6,0,18)]\ .$$

As X^3_{21} satisfies feasibility in $AX = b$, it follows that

$$X^3_{(2)} \cap S \neq \phi$$

$\therefore X^3_{21}$ is the optimal solution.
$\therefore$ Optimal solution of the original problem is

$$x_1 = 5,\ x_2 = 0$$

and optimal value is 5.

References

Balimski, M. L.: An Algorithm for Finding all Vertices of Convex Polyhedron Sets, f. Sec. Indust. Appl. Maths., Vol. 9, 1, March 61, pp. 702–788.

Cernikova, N. V.: An Algorithm for the determination of the set of all solutions of a linear programming problem, Akademija Nauk SSSR Žurnal Vyčislitel naî Matematika i Mathemati cesksĭ Fiziki (Moscow), Vol. 8, 1968, pp. 1387–1395.

Hadley: Linear Programming, Addison-Wesley, Reading, Mass, 1962.

Kirby, M. J. L., H. R. Love, and *Kanti Swarup,* Extreme Point Mathematical Programming, Management Science, May 1972, U.S.A.

–: A cutting plane algorithm for Extreme point Mathematical Programming, Cahier D'Etudes de Recherche Oper', 1971. Belgium.

– : An Enumeration Technique for Extreme Point Mathematical Programming Problems, Research Report, 1970, Dulhousie University, Canada.

Puri, M. C., and *Kanti Swarup:* Deep cut Procedure for Extreme Point Mathematical Programming Problems, to appear in 'Cahier D'Etudes de Recherche Oper', 1973, Belgium.

– : On Enumerative Procedure for Extreme Point Mathematical Programming Problems, submitted.