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Ranking in Integer Linear Fractional Programming Problems

By V. Verma[1], H.C. Bakhshi[2] and M.C. Puri[1]

Abstract: An algorithm is developed which ranks the feasible solutions of an integer fractional programming problem in decreasing order of the objective function values.

Zusammenfassung: Es wird ein Algorithmus angegeben, der die zulässigen Lösungen eines ganzzahligen Quotientenprogrammes nach fallenden Zielfunktionswerten liefert.

Keywords: integer fractional programming, ranking

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An algorithm is developed to rank the various integer feasible solutions of an integer linear fractional programming problem in decreasing order of values of objective function.

Integer linear fractional programming problems have been solved by many authors [1, 2, 3, 4, 5]. These problems find their applications in quite a large number of areas like fixed-charge problems, job-shop scheduling problems. Many times optimal integer feasible solutions of these problems are not of practical interest due to certain limitations. In such situations, one is interested in ranking the integer feasible solutions in decreasing order of values of objective function and from the exhaustive list of various integer solutions with distinct values of objective function, one can select that solution which is most suitable under the existing limitations. Ranking of integer solutions is also useful in bicriteria linear fractional programming problems and multicriteria linear fractional programming problems.

A possible approach for ranking integer feasible solutions of integer linear fractional programming problems could be to use Dantzig-cut. This may however be a long process, firstly because of its slow convergence rate and second

[1] Vanita Verma and M.C. Puri, Department of Mathematics, Indian Institute of Technologiy, Hauz Khas, New Dehli – 110016, India
[2] H.C. Bakhshi, Motilal Nehru College, Dehli University, Dehli

because after obtaining any kth best integer solution, it yields $(k + 1)$th best integer solution, only after computing all the integer solutions alternate to the kth best integer solution.

In integer linear programming problem [6], this situation can be handled by introducing cuts defined by $cX \leq Z^k - 1$, where cX is the linear objective function to be maximized and Z^k is the kth best objective function value. Optimal solution obtained over the truncated region is the desired $(k + 1)$th best integer solution. This cut apart from being deeper than Dantzig-cut, avoids scanning all the alternates at any stage. Unfortunately, this approach is not applicable to integer linear fractional case, since decrease in objective function may not be one, whereas in linear case, decrease is at least one. In the present paper, a cut is developed, which helps in ranking the integer solutions of an integer linear fractional programming problem without having to evaluate all integer solutions alternate to a known integer solution.

The integer linear fractional programming problem can be stated as:

$$(P)\quad \text{Maximize} \quad Z = \frac{cX + \alpha}{dX + \beta}$$

$$\text{subject to} \quad X \in F$$

$$x_j \geq 0 \text{ and integers for all } j$$

where F is a closed and bounded convex polyhedron over which $dX + \beta > O$, c, $d, X \in R^n$, $\alpha, \beta \in R$.

The following notations have been defined

c_j: jth component of vector c
d_j: jth component of vector d
X^k: a kth best integer feasible solution of (P)

X^k will be an extreme point of $S(k)$ which is obtained from F after its successive truncations.

Let $S(k) = \{X \mid AX = b, X \geq 0\}$, where $A \in R^{m \times n}$, $b \in R^m$.

B^k: Basis corresponding to X^k
$I(k)$: $\{i \mid x_i$ is a basic variable in the Simplex tableau corresponding to $X^k\}$

Let $T(k) = \overline{I(k)} - J(k)$, were $\overline{I(k)}$ is defined as $\overline{I(k)} = \{j \mid x_j$ is a non-basic variable in the simplex tableau corresponding to $X^k\}$

$$n^k = cX^k + \alpha$$
$$d^k = dX^k + \beta$$
$$Z(X^k) = n^k / d^k$$

For $j \in \overline{I(k)}$, define the following

$$Z_j^1 = \sum_{i \in I(k)} c_i y_{ij}$$

where $a_j = \sum_{i \in I(k)} a_i y_{ij}$, i.e. $y_{ij} = ith$ component of $(B^k)^{-1} a_j$

$$Z_j^2 = \sum_{i \in I(k)} d_i y_{ij}$$
$$\Delta_j^k = n^k(Z_j^2 - d_j) - d^k(Z_j^1 - c_j)$$
$$J(k) = \{j \mid j \in \overline{I(k)},\ \Delta_j^k = 0\}$$

In order to develop the cut, following results are established.

Theorem: All integer solutions of (P) yielding value $Z(X^k)$ lie in the closed half space

$$\sum_{j \in J(k)} x_j \geq 1 .$$

Proof: Since X^k is an integer feasible solution of problem (P),

$$\sum_{j \in I(k)} a_j x_j = b .$$

Therefore, $\sum_{j \in I(k)} a_j x_j - \phi_j a_j + \phi_j a_j = b$, for some $j \in J(k)$, where ϕ is a non-zero scalor.

$$\Rightarrow \sum_{i \in I(k)} a_i x_i - \phi_j \sum_{i \in I(k)} a_i y_{ij} + \phi_j a_j = b$$
$$\Rightarrow \sum_{i \in I(k)} a_i (x_i - \phi_j y_{ij}) + \phi_j a_j = b \qquad (1)$$

If $0 < \phi_j \leq \min_{i \in I(k)} \left\{ \frac{x_i}{y_{ij}} \mid y_{ij} > 0 \right\}$, then (1) implies that

$$\hat{X}^k = \begin{cases} \hat{x}_i = x_i - \phi_j y_{ij},\ i \in I(k) \\ \hat{x}_j = \phi_j,\ j \in J(k) \\ \hat{x}_t = 0,\ t \in \overline{I\ (k)} - \{j\} \end{cases}$$

is a new integer feasible solution, provided ϕ_j is a positive integer and $\phi_j y_{ij}$ is an integer for every $i \in I\ (k)$.

Consider $Z(\hat{X}^k) - Z(X^k) = \dfrac{\sum\limits_{i \in I(k)} c_i \hat{x}_i + c_j \hat{x}_j + \alpha}{\sum\limits_{i \in I(k)} d_i \hat{x}_i + d_j \hat{x}_j + \beta} - \dfrac{n^k}{d^k}$

$$= \frac{\sum\limits_{i \in I(k)} c_i (x_i - \phi_j y_{ij}) + c_j \phi_j + \alpha}{\sum\limits_{i \in I(k)} d_i (x_i - \phi_j y_{ij}) + d_j \phi_j + \beta} - \frac{n^k}{d^k}$$

$$= \frac{n^k - \phi_j (Z_j^1 - c_j)}{d^k - \phi_j (Z_j^2 - d_j)} - \frac{n^k}{d^k}$$

$$= \frac{\phi_j \Delta_j^k}{d^k [d^k - \phi_j (Z_j^2 - d_j)]} = 0 (\because j \in J(k))$$

$\Rightarrow \hat{X}^k$ is alternate to X^k .

Clearly $\hat{X}^k$ satisfies $\sum\limits_{j \in J(k)} x_j \geq 1$.

Similarly each of the other possible integer solutions alternate to X^k, which may lie in directions of other $x_j,\ j \in J(k)$ would satisfy $\sum\limits_{j \in J(k)} x_j \geq 1$.

Let X^* be an extreme point solution of $S(k)$ alternate to integer feasible solution X^k . X^* may not have all integer components. Let X^* be derived from X^k by entering x_q ($q \in J(k)$) and departing x_r ($r \in I(k)$). Then for a $j \in J(k)$ such that x_j is still non-basic in the Simplex tableau corresponding to X^*,

$$\Delta_j^* = n^* \left[(Z_j^2 - d_j) - \frac{y_{r_j}}{y_{r_q}} (Z_q^2 - d_q) \right] - d^* \left[(Z_j^1 - c_j) - \frac{y_{r_j}}{y_{r_q}} (Z_q^1 - c_q) \right]$$

$$= \frac{d^*}{d^k} \left[\Delta_j^k - \frac{y_{r_j}}{y_{r_q}} \Delta_q^k \right] \left(\because Z(X^*) = \frac{n^*}{d^*} = \frac{n^k}{d^k} \right)$$

$$= 0 (\because j, q \in J(k)) .$$

Similary for a $t \notin J(k)$ such that x_t is still non - basic in X^*,

$$\Delta_t^* = \frac{d^*}{d^k} \Delta_t^k < 0$$

$$\Delta_y^* = \frac{d^*}{d^k} \Delta_r^k = 0 \; (\because x_r \text{ is basic in } X^k)$$

(Now x_r is non-basic in X^*)

Thus $\Delta_j^* = 0$ for all $j \in J(k)$, x_j is non - basic in X^*

$$\Delta_r^* = 0$$

$$\Delta_t^* < 0 \text{ for all } t \notin J(k), x_t \text{ is non - basic in } X^*.$$

Clearly for an integer feasible solution alternate to X^* and hence alternate to X^k, if it exists, on an edge (other than joining X^k and X^*) originating from X^*, there must exist at least one $j \in J(k)$, $(j \neq q)$ such that $x_j \geq l$ and integer. Thus such like integer feasible solutions alternate to X^k also satisfy $\sum_{j \in J(k)} x_j \geq 1$.

Consider an integer solution W^k, (if it exists), which is convex combination of integer feasible solutions alternate to X^k i.e.,

$$W^k = \lambda^k X^k + \hat{\lambda}^k \hat{X}^k + \ldots, \lambda^k + \hat{\lambda}^k + \ldots = 1$$

$$\lambda^k \geq 0, \hat{\lambda}^k \geq 0, \ldots$$

Also,

$$Z(W^k) = \frac{cW^k + \alpha}{dW^k + \beta}$$

$$= \frac{c(\lambda^k X^k + \hat{\lambda}^k \hat{X}^k + \ldots) + \alpha(\lambda^k + \hat{\lambda}^k + \ldots)}{d(\lambda^k X^k + \hat{\lambda}^k \hat{X}^k + \ldots) + \beta(\lambda^k + \hat{\lambda}^k + \ldots)}$$

$$= \frac{\lambda^k (cX^k + \alpha) + \hat{\lambda}^k (c\hat{X}^k + \alpha) + \ldots}{\lambda^k (dX^k + \beta) + \hat{\lambda}^k (d\hat{X}^k + \beta) + \ldots}$$

$$= Z(X^k) \frac{(\lambda^k d^k + \hat{\lambda}^k \hat{d}^k + \ldots)}{(\lambda^k d^k + \hat{\lambda}^k \hat{d}^k + \ldots)} \left(\because Z(X^k) = \frac{n^k}{d^k} = \frac{\hat{n}^k}{\hat{d}^k} \right)$$

$$= Z(X^k).$$

As the closed half space $\sum_{j \in J(k)} x_j \geq 1$ is a convex set and X^k, $\hat{X}^k$,... all belong to it, W^k also belongs to it. Therefore, W^k also satisfies

$$\sum_{j \in J(k)} x_j \geq 1.$$

Corollary: All integer solutions of (P) yielding value $Z(X^k)$ of the objective function lie in the open half space

$$\sum_{j \in T(k)} x_j < 1$$

Proof follows from the fact that for every integer feasible solution of (P) alternate to X^k, $x_j = 0$ for all $j \in \overline{I\,(k)} - J(k)$.

Theorem 2: All integer solutions of (P) yielding value less than $Z(X^k)$ lie in the closed half space

$$\sum_{j \in T(k)} x_j \geq 1.$$

Proof: From an optimal integer feasible solution X^k over $S(k)$ at the kth stage, a solution $\overline{X}^k$ can be derived by increasing x_j, $j \in \overline{I\,(k)} - J(k)$ upto ϕ_j (> 0). As in Theorem 1, $\overline{X}^k$ will be given as:

$$\overline{X}^k = \begin{cases} \overline{x}_i = x_i - \phi_j y_{ij}, & i \in I(k) \\ \overline{x}_j = \phi_j, & j \in \overline{I\,(k)} - J(k) \\ \overline{x}_v = 0, & v \in \overline{I\,(k)} - \{j\} \end{cases}$$

$\overline{X}^k$ is an integer feasible solution provided

(i) $\phi_j \leq \min_{i \in I(k)} \left\{ \frac{x_i}{y_{ij}} \mid y_{ij} > 0 \right\}$

(ii) ϕ_j is a positive integer, and

(iii) $\phi_j y_{ij}$ is an integer for all $i \in I(k)$.

Since ϕ_j is a positive integer, $\overline{X}^k$ satisfies $\sum_{j \in T(k)} x_j \geq 1$.

Consider $Z(\overline{X}^k) - Z(X^k) = \dfrac{\sum_{i \in I(k)} c_i \overline{x}_i + c_j x_j + \alpha}{\sum_{i \in I(k)} d_i \overline{x}_i + d_j x_j + \beta} - \dfrac{n^k}{d^k}$

$$= \frac{\sum\limits_{i\in I(k)} c_i(x_i - \phi_j y_{ij}) + c_j\phi_j + \alpha}{\sum\limits_{i\in I(k)} d_i(x_i - \phi_j y_{ij}) + d_j\phi_j + \beta} - \frac{n^k}{d^k}$$

$$= \frac{n^k - \phi_j(Z_j^1 - c_j)}{d^k - \phi_j(Z_j^2 - d_j)} - \frac{n^k}{d^k}$$

$$= \frac{\phi_j \Delta_j^k}{d^k[d^k - \phi_j(Z_j^2 - d_j)]} < 0 (\because \Delta_j^k < 0$$

for $j \in \overline{I(k)} - j(k))$

Hence $Z(\overline{X}^k) < Z(X^k)$

Similarily, all other integer feasible solutions which can be derived from X^k by moving in direction x_j, $j \in \overline{I(k)} - J(k)$ will yield value less than $Z(X^k)$ and lie in the closed half space $\sum\limits_{j\in T(k)} x_j \geq 1$. Otherwise, also, in any integer feasible solution yielding value less than $Z(X^k)$, there exists at least one $j \in \overline{I(k)} - J(k)$, for which $x_j \geq 1$ and integer. Thus that solution lies in the closed half space $\sum\limits_{j\in T(k)} x_j \geq 1$.

Algorithm

Initial Step: Find X^1 for the integer linear fractional programming problem (P).

General Step: $(k \geq 1)$

Introduce the cut $\sum\limits_{j\in T(k)} x_j \geq 1$ and solve the problem:

$$(P^k) \qquad \text{Maximize} \qquad Z = \frac{cX + \alpha}{dX + \beta}$$

$$\text{subject to} \qquad X \in S(k)$$

$$\sum_{j\in T(k)} x_j \geq 1$$

$$x_j \geq 0$$

and integers for all j.

Optimal solution of (P^k) gives x^{k+1}.

Termination: The list of ranked integer feasible solutions of (P) is complete, when continuing the general step, we reach the problem (P^q):

$$(P^q) \qquad \text{Maximize} \qquad Z = \frac{cX + \alpha}{dX + \beta}$$

$$\text{subject to} \qquad X \in S(q)$$

$$\sum_{j \in T(q)} x_j \geq 1$$

$$x_j \geq 0$$

and integer for all j,

whose set solution is empty. The process is terminated then.

Remark 1: If the problem (P^q) has no feasible solution, then there are q distinct values of the objective function ranked in decreasing order.

Remark 2: The procedure presented in this paper obtains all distinct values of the objective function at various integer feasible solutions in a finite number of steps because

(i) at any stage (say at kth stage), introduction of the cut $\sum_{j \in T(k)} x_j \geq 1$ yields an integer feasible solution giving lesser value of objective function than that at X^k and reduction is finite.

(ii) integer solutions alternate to some integer solution say X^k, are discarded by the cut $\sum_{j \in T(k)} x_j \geq 1$ and hence don't appear in computations.

Concluding Remarks

Since we are unaware of any technique which could rank integer feasible solutions in order of decreasing values of objective function in an integer linear fractional programming problem, our main aim is to develop an algorithm to achieve the same. The proposed method is computationally efficient, since the proposed cut $\sum_{j \in T(k)} x_j \geq 1$ is better than Dantzig cut, as it discards all alternate integer solutions yielding same value of objective function and yields next best integer solution, with a definite decrease in value of objective function. Also it is easy to implement the proposed cut (since to derive X^{k+1} from X^k, one has just to append the cut in the Simplex table corresponding to X^k and carry out pivoting iterations as in an ordinary integer linear fractional programming problem). Also at each step, a cut which becomes inactive can be dropped.

Numerial

$$(P) \quad \text{Maximize} \quad Z = \frac{6x_1 + 6x_2}{11x_1 + x_2 + 5}$$

$$\text{subject to} \quad x_1 \le 4$$

$$2x_2 \le 7$$

$$x_1, x_2 \ge 0$$

and integers.

Solving the problem (P) we get $X^1 = (X_1, x_2) = (0, 3)$ and $Z(X^1) = 2.25$. Introducing the cut $\sum_{j \in T(1)} x_j \ge 1$, solve the problem (P^1) to get $X^2 = (x_1, x_2) = (0,2)$ and $Z(X^2) = 1.71428$.

Similarily solving problems (P^2), (P^3) and so on, we get the complete list of integer feasible solutions of (P) ranked in order of decreasing value of objective function as:

i	$X^i = (x^1, x^2)$	$Z(X^i)$
1	(0,3)	2.25
2	(0,2)	1.71428
3	(1,3)	1.26315
4	(0,1)	1.00000
5	(3,3)	0.87804
6	(2,2)	0.82758
7	(4,3)	0.80769
8	(3,2)	0.75000
9	(4,2)	0.70588
10	(2,1)	0.64285
11	(3,1)	0.61538
12	(4,1)	0.60000
13	(4,0)	0.48979
14	(3,0)	0.47368
15	(2,0)	0.44444
16	(1,0)	0.37500
17	(0.0)	0.00000

Infeasibility is declared when (P^{17}) is reached (i.e., when the cut $\sum_{j \in T(17)} x_j \ge 1$ is introduced), indicating that all integer feasible solutions, giving distinct values of objective function have been scanned.

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