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A Min Max Problem

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Abstract: This paper studies a special class of min max problems in two sections. In Section I, a procedure is developed which gives the optimal solution of the problem. The Section II deals with ranking the solutions in increasing order of the value of the objective function.

Zusammenfassung: Eine spezielle Klasse von Minimax-Problemen wird untersucht. In Teil I wird ein Verfahren zur Bestimmung der Optimal-Lösung des Problems entwickelt. Teil II behandelt die Anordnung der Lösungen entsprechend ansteigender Werte der Zielfunktion.

The class of min max problem that we propose to study is

$$\underset{X \in S}{\text{Min}} \; [\underset{x_j \in X}{\text{Max}} \; f_j(x_j)] \tag{P}$$

where x_j is the j-th component of X and S is the given convex polyhedron $\{X \mid AX = b, X \geqslant 0\}$, the matrix A being of order $m \times n$ and

$$f_j(x_j) = f_j > 0 \qquad \text{if } x_j > 0$$
$$= 0 \qquad \text{if } x_j = 0.$$

The min max problem (P) finds its application in time minimization transportation problem [*Garfinkel/Rao; Hammer*] and also in situations where the fixed costs or set up costs incurred by various producers in setting up different factories to produce a given number of products independently (one producer manufacturing one product only) is higher than what it would be if a single producer were to produce all the products. In the former case the fixed cost is $\sum_{x_j \in X} f_j(x_j)$ and then the problem would be $\underset{X \in S}{\text{Min}} \; [\sum_{x_j \in X} f_j(x_j)]$. He would be able to save on building costs, wages of managers, in-

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surance charges, etc. In such situations we assume that the fixed cost or set up cost of such a producer is $\underset{x_j \in X}{\text{Max}}\ f_j(x_j)$, (Mostly it is a lower bound on the fixed cost/set up cost).

Section I

Theorem 1: The function $F(X) = \underset{x_j \in X}{\text{Max}}\ f_j(x_j)$ is a concave function.

Proof: Let X^1, X^2 be any two points of S.

Let $F(X^1) = F^1$, $F(X^2) = F^2$, $F^1 \geqslant F^2$.

Let $X = \lambda X^1 + (1-\lambda)X^2, \ \ 0 \leqslant \lambda \leqslant 1.$

Case (i). $0 < \lambda \leqslant 1$.

Let the i-th component x_i^1 of X^1 yield the value F^1 of $F(X^1)$. Then the i-th component $\lambda x_i^1 + (1-\lambda)x_i^2$ of X is positive.

$$\begin{aligned} F(X) &= F(\lambda X^1 + (1-\lambda)X^2) \\ &= F^1 \\ &= \lambda F^1 + (1-\lambda)F^1 \\ &\geqslant \lambda F^1 + (1-\lambda)F^2 \\ &= \lambda F(X^1) + (1-\lambda)F(X^2). \end{aligned}$$

Case (ii). $\lambda = 0$.

$$\begin{aligned} F(X) &= F(X^2) \\ &= F^2 \\ &= \lambda F^1 + (1-\lambda)F^2 \\ &= \lambda F(X^1) + (1-\lambda)F(X^2). \end{aligned}$$

Hence in all cases

$$F(\lambda X^1 + (1-\lambda)X^2) \geqslant \lambda F(X^1) + (1-\lambda)F(X^2).$$

Thus, $F(X)$ is a concave function.

Corollary: The global minimum of $F(X)$ occurs at an extreme point of S.

Proof:

(i) Follows directly from the Theorem 1 [*Mangasarian*].

(ii) Let X be a non basic feasible solution of the problem (P), and let $F(X) = F$. From this feasible solution X, a basic feasible solution $\hat{X}$ can be derived by reducing some of the positive components in X to zero [*Hadley*]. This implies that

$$\max_{\hat{x}_j \in \hat{X}} f_j(\hat{x}_j) \leqslant \max_{x_j \in X} f_j(x_j)$$

i.e.,

$$F(\hat{X}) \leqslant F(X).$$

Thus, $\hat{X}$ is atleast as good as X is.

Therefore, the global minimum will be attained at atleast one basic feasible solution of (P).

This motivates one to study only the basic feasible solutions of (P).

Let $\bar{X} = [\bar{x}_j]$ be any basic feasible solution of the problem (P) and let $F(\bar{X}) = \max_{\bar{x}_j \in \bar{X}} f_j(\bar{x}_j) = \bar{F}$. With each variable x_j, associate a cost c_j as follows:

$$\begin{aligned} c_j &= 0 \quad \text{if } f_j < \bar{F} \\ &= 1 \quad \text{if } f_j = \bar{F} \\ &= \infty \quad \text{if } f_j > \bar{F}. \end{aligned}$$

Then the cost minimization linear programming problem

$$\begin{aligned} \text{Min} \quad & z(X) = \Sigma c_j x_j \\ \text{subject to} \quad & AX = b \qquad (\text{LP}(\bar{X})) \\ & X \geqslant 0 \end{aligned}$$

is associated with each basic feasible solution $\bar{X}$ of (P).

Definitions

1. *Better solution*: Let $X^* = [x_j^*]$ and $X^{**} = [x_j^{**}]$ be any two basic feasible solutions of the problem (P) giving $F(X^*) = F^*$, $F(X^{**}) = F^{**}$. Then X^{**} is said to be a better solution than X^* if

either (i) $F^{**} < F^*$

or (ii) $F^{**} = F^*$ but $\sum_{\{j \mid f_j = F^{**}\}} x_j^{**} < \sum_{\{j \mid f_j = F^*\}} x_j^*$.

That is, a better solution is one which involves either (i) lesser set up cost or (ii) the same set up cost but involves lesser number of units of various products manufactured at that very set up cost.

2. *Optimal solution*: A basic feasible solution X is said to be an optimal solution of (P) if there does not exist any other solution better than X.

Theorem 2: A local minimum solution is also global minimum solution.

Proof: Let $X_1 = [x_j']$ be a local minimum basic feasible solution of the problem (P) giving $F(X_1) = F_1$.
Suppose X_1 is not a global minimum solution.
Therefore, there exists a better basic feasible solution $X_2 = [x_j^2]$ of the problem (P) giving $F(X_2) = F_2$.
By definition (1),

(i) $F_2 < F_1$

or

(ii) $F_2 = F_1$ but $\sum_{\{j|f_j=F_2\}} x_j^2 < \sum_{\{j|f_j=F_1\}} x_j^1$.

Consider the cost minimization linear programming problem (LP (X_1)) associated with X_1

$$\begin{aligned} &\min z(X) = \Sigma c_j x_j \\ &\text{subject to } AX = b \qquad\qquad (\text{LP}(X_1)) \\ &\qquad\qquad X \geqslant 0 \end{aligned}$$

where

$$\begin{aligned} c_j &= 0 \quad \text{if } f_j < F_1 \\ &= 1 \quad \text{if } f_j = F_1 \\ &= \infty \quad \text{if } f_j > F_1 . \end{aligned}$$

Note that in case (ii), when $F_2 = F_1$, (LP (X_1)) is same as (LP (X_2)) and $z(X_2) < z(X_1)$.
Clearly, $z(X_1)$ is nonzero finite.
In case (i), since $F_2 < F_1$, $z(X_2) = 0 < z(X_1)$.
In case (ii) also, $z(X_2) < z(X_1)$.
Thus, in both the cases, X_2 is a solution of (LP (X_1)) better than X_1.
Therefore, X_1 is not an optimal solution of (LP (X_1)).
Thus, there exists an adjacent basic feasible solution X_3 (say) of LP (X_1) better than X_1.

$\therefore z(X_3) < z(X_1)$

Let $F(X_3) = F_3$.

Clearly $F_3 \not> F_1$.

$\therefore$ By definition, X_3 is a basic feasible solution of (P) adjacent to X_1 and better than X_1 which contradicts that X_1 is a local minimum solution of (P).
Hence the Theorem.

Procedure

Let $X^{(1)}$ be an initial basic feasible solution of the problem (P), giving $F(X^{(1)}) = F^{(1)}$.
Solve the associated cost minimization linear programming problem (LP $(X^{(1)})$). Let $X^{(2)}$ be its optimal basic feasible solution yielding the value of the objective function as $z(X^{(2)})$.

Case (i): $z(X^{(2)})$ is nonzero finite.
This implies that there does not exist a solution of (P) in which all the variables x_j for which $f_j(x_j) = F^{(1)}$ are zero. Therefore, $X^{(2)}$ is the optimal solution of (P).

Case (ii): $z(X^{(2)}) = 0$
Let $F(X^{(2)}) = F^{(2)}$
Obviously, $F^{(2)} < F^{(1)}$.

Solve (LP $(X^{(2)})$) and let $X^{(3)}$ be an optimal basic feasible solution of (LP $(X^{(2)})$).

If $z(X^{(3)})$ for (LP $(X^{(2)})$) is zero, continue solving the associated problems, (LP $(X^{(3)})$), (LP $(X^{(4)})$), . . . , till a problem (LP $(X^{(k)})$) is reached which has nonzero finite optimal value $z(X^{(k+1)})$ of the objective function, where $X^{(i)}$ is an optimal basic feasible solution of (LP $(X^{(i-1)})$).

Thus, case (i) is reached and $X^{(k+1)}$ is an optimal solution of (P).

The process converges in a finite number of steps as only basic feasible solutions of (P) are studied which are always finite in number. Also as at each stage, solution is improved, cycling will never appear.

Section II

In some situations, it may be disadvantageous to accept the minimum set up cost solution and naturally one is forced to study the acceptability of the next minimum se up cost solution (i.e., the minimum set up cost solution after excluding the minimal set up cost solution). For example, caring too much for set up cost sometimes leads to a poor quality of products on account of less sophisticated machinery used and such products do not find their market. In such situations one has to switch over to better machinery which incurres set up cost higher than the minimum possible set up cost. Also there may be situations where the minimum set up cost solution may involve a formidable time of completion of the project. Such situations again call for better machinery for faster production and this involves a higher set up cost. In general, one may be interested in k-th minimal set up cost solution, $k = 1, 2, \ldots$ To determine these solutions in order of increasing set up cost, we modify and apply *Murty*'s approach [*Murty*].

Some Definitions

Let T be the set of all basic feasible solutions in S.

Node

A node N is a set of elements of T whose bases do not contain some specified columns of A.

Let $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ be specified columns of A which do not belong to the bases of elements in N. Then we write N as

$$N = (\bar{a}_{i_1}, \bar{a}_{i_2}, \ldots, \bar{a}_{i_k})$$

The min max problem at node N

Suppose node N is given by

$$N = (\bar{a}_{i_1}, \bar{a}_{i_2}, \ldots, \bar{a}_{i_k}).$$

Replacing f_i's for $i = i_1, i_2, \ldots, i_k$ by a large positive number say, M, which is greater than any number with which it is compared, the resulting problem

$$\text{Min } F(X) = \underset{x_j \in X}{\text{Max}} \; f_j(x_j)$$

$$\text{subject to} \quad AX = b$$

$$X \geqslant 0$$

is called as the min max problem at node N.

The optimal solution of the min max problem at node N – is denoted by X_N.

Obviously, the optimal solution X_N will be in N if $F(X_N)$ is independent of M, otherwise N will be an empty set.

Covering node N by its optimal solution X_N

Let the optimal solution of the min max problem at node N be X_N and its basis be B_N, where

$$B_N = (b_1, b_2, \ldots, b_m)$$

each of b_i is different from $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$. Then N can be expressed as the union of $\{X_N\}$ and the node $N_1, N_2, \ldots, N_m$ where

$$N_j = (\bar{a}_{i_1}, \bar{a}_{i_2}, \ldots, \bar{a}_{i_k}, \bar{b}_j), \quad j = 1, \ldots, m.$$

By definition, none of $N_1, N_2, \ldots, N_m$ contains X_N and

$$N = \{X_N\} \cup \bigcup_{i=1}^{m} N_i.$$

This operation will be called the covering of node N by its optimal solution X_N.

Procedure

Determine the optimal solution of the min max problem (P). Let it be $X_{(1)}$ given by the basis

$$B_{(1)} = (b_1, b_2, \ldots, b_m)$$

and $F(X_{(1)}) = F_{(1)}$.

Define the nodes $N_1, N_2, \ldots, N_m$ by

$$N_j = (\bar{b}_j), \quad j = 1, \ldots, m.$$

Solve the min max problems at nodes N_j, $j = 1, \ldots, m$.

Let

$$F_{(2)} = \min_{j=1 \text{ to } m} F(X_{N_j}) = F(X_{N_d}) \quad \text{(say)}$$

where X_{N_j} denotes the optimal solution of the min max problem at node N_j.

Clearly $F_{(1)} \leqslant F_{(2)}$.

Cover the node N_d by its optimal solution X_{N_d}.

Let

$$N_d = \{X_{N_d}\} \cup \bigcup_{i=1}^{m} N_{d_i}.$$

Solve the min max problems at nodes N_{d_i}, $i = 1, \ldots, m$.

Let

$$F_{(3)} = \min_{\{j | j \neq d, j=1, \ldots, m\} \cup \{j | j = d_i, i=1, \ldots, m\}} F(X_{N_j}) = F(X_e) \quad \text{(say)}.$$

Then

$$F_{(2)} \leqslant F_{(3)}.$$

Covering the node N_e by its optimal solution X_e and continuing, determine $F_{(4)}$, $F_{(5)}$ etc.

If $F_{(2)} = F_{(1)}$, X_{N_d}, is alternative optimal solution of the problem (P). If $F_{(2)} > F_{(1)}$, then X_{N_d} is 2nd best basic feasible solution of the problem (P).

Let $F_{(1)} < F_{(2)} < \ldots < F_{(k)}$ be the ranking of the values of the objective function up to k-th best basic feasible solution given by $X_{(1)}, X_{(2)}, \ldots, X_{(k)}$ respectively.

Let F' be the next value of objective function of the problem (P) as determined above, given by the basic feasible solution X'.

If $F' = F_{(k)}$, then X' is alternate k-th best solution.
If $F_{(k)} < F'$, then X' is $(k+1)$-th best solution.

If at a stage, $\underset{j}{\text{Min}}\, F(X_{N_j}) = M$, i.e., each of the nodes N_j is empty, the process ends and the preceding stage gives the last value of the objective function of the problem (P).

Remark

If the slack or surplus variables are added to constraints to reduce them in the form $AX = b$, then f_j's associated with them are taken to be max (f_j). This helps in obtaining a basic feasible solution independent of them, if there is any. If these variables appear in optimal basic feasible solution, then the optimal value is read from the original variables.

Example

$$\text{Min}\, F(X) = \underset{x_j \in X}{\text{Max}}\, f_j(x_j)$$

$$\begin{aligned} \text{subject to}\quad 3x_1 - 2x_2 - x_4 - 7x_5 + 2x_6 + x_7 &= 1 \\ 2x_1 + 2x_3 + x_4 - 5x_5 \qquad\quad + x_7 &= 4 \\ x_2 + 2x_3 + x_4 - 6x_5 + x_6 + x_7 &= 3 \\ x_i \geqslant 0,\ i = 1, \ldots, 7. \end{aligned}$$

where

$$\begin{aligned} f_j(x_j) &= f_j \quad \text{if } x_j > 0 \\ &= 0 \quad \text{if } x_j = 0 \end{aligned}$$

and f_i's are given by 12, 9, 14, 4, 7, 6, 2 respectively.
For this numerical example

$$X^{(1)} = (17/3, 0, 0, 9, 1, 0, 0),\ F(X^{(1)}) = 12.$$

Solving (LP ($X^{(1)}$)), the optimal solution $X^{(2)}$ is given by

$$X^{(2)} = (0, 1/2, 0, 0, 1/2, 0, 11/2), F(X^{(2)}) = 9.$$

Solving (LP ($X^{(2)}$)), the optimal solution $X^{(3)}$ is given by

$$X^{(3)} = (0, 0, 0, 1/2, 1, 0, 17/2), F(X^{(3)}) = 7.$$

Since (LP ($X^{(3)}$)) has non-zero finite optimal value 1, $X^{(3)}$ is optimal solution for the problem (P).

Ranking

The optimal solution of the min max problem (P) is

$X_{(1)} = (0, 0, 0, 1/2, 1, 0, 17/2)$, yielded by the basis

$B_{(1)} = (a_4, a_5, a_7)$ and $F(X_{(1)}) = F_{(1)} = 7$.

Defining the nodes N_1, N_2, N_3 by

$$N_1 = (\bar{a}_4), N_2 = (\bar{a}_5),\ N_3 = (\bar{a}_7)$$

and solving the min max problems at nodes N_1, N_2, N_3, we get

$$X_{N_1} = (0, 1/2, 0, 0, 1/2, 0, 11/2), F(X_{N_1}) = 9$$
$$X_{N_2} = (1/3; 1, 0, 0, 0, 0, 2), F(X_{N_2}) = 12$$
$$X_{N_3} = (1, 2, 0, 0, 0, 1, 0), F(X_{N_3}) = 12$$
$$F_{(2)} = \min_{i=1,2,3} F(X_{N_i}) = 9.$$

Since $7 < 9, F_{(2)} = 9$ is the 2nd best value of $F(X)$ and $X_{N_1} = (0, 1/2, 0, 0, 1/2, 0, 11/2)$ is the 2nd best basic feasible solution of (P).

Covering the node N_1 by its optimal solution X_{N_1} and solving the min max problems at nodes thus obtained, we get

$$F_{(3)} = 12.$$

Continuing as above, the other higher values of $F(X)$ can be obtained.

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