

## Extreme Point Linear Fractional Functional Programming

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Eingegangen am 27. Februar 1973

*Abstract:* This paper deals with the optimization of the ratio of two linear functions subject to a set of linear constraints with the additional restriction that the optimal solution is to be an extreme point of another convex polyhedron. In this paper, an enumerative procedure for solving such type of problems is developed. For an illustration, a numerical example is also provided.

### Introduction

Very recently a lot of work has been done in extreme point linear programming [Kirby, Love and Swarup, 1972, and 1970] wherein a linear objective function has been optimized over a convex polyhedron with the additional requirement that the optimal value should exist on an extreme point of another convex polyhedron. An extreme point technique has also been used in solving the Fixed Charge Problem and the Assignment Problem [Puri and Swarup, 1973/74, and Puri, w. y.]. Here, the objective function to be optimized is the ratio of two linear functions. This objective function which is neither convex nor concave, is optimized subject to linear inequalities with the additional constraint that the optimal solution should also be an extreme point of another convex polyhedron.

Some practical problems [Swarup, 1965, and Martos, 1964] of this type are zero-one integer fractional programming problem in which the requirement that each of the variables be either zero or one can be replaced by the condition that the optimal solution be an extreme point of the unit cube  $IX \leq 1, X \geq 0$ . Extreme point fractional functional programming problem is a larger class of problems than the class of zero-one integer fractional programming problems.

The most general mathematical form of extreme point linear fractional functional programming problem is:

$$\begin{array}{l}
 \text{Max } Z = \frac{CX + \alpha}{DX + \beta} \\
 \text{subject to } AX = b \\
 \text{and } X \text{ is an extreme point of} \\
 \quad \quad \quad RX = t \\
 \quad \quad \quad X = 0
 \end{array}
 \quad \left| \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right.
 \quad \dots \text{ Problem (I)}$$

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where  $C, D$  are  $1 \times n$ ,  $X$  is  $n \times 1$ ,  $A$  is  $m \times n$ ,  $R$  is  $p \times n$ ,  $b$  is  $m \times 1$ ,  $t$  is  $p \times 1$ ,  $0$  is  $n \times 1$  and  $\alpha, \beta$  are scalar constants.

The problem (I) is always bounded as any solution is to be an extreme point of  $RX = t, X \geq 0$ . We shall develop an enumerative technique in which extreme points of the convex polyhedron  $AX = b, RX = t, X \geq 0$  are enumerated till we reach the required optimal solution. As the number of these extreme points is finite, the process will converge in a finite number of steps. When (I) has no solution, an indication to this effect will appear in the algorithm itself. The algorithm developed will be a dual method type of algorithm in the sense that feasibility and optimality of (I) are reached simultaneously.

The remaining sections of this paper contain the theory of the method for solving (I) and an algorithm based on this theory.

*Theoretical Development:*

To solve (I) we formulate another problem:

$$\begin{aligned} \text{Max } Z &= \frac{CX + \alpha}{DX + \beta} & | & \\ \text{subject to } FX &= f & | & \\ & X \geq 0 & | & \end{aligned} \quad \dots \text{ Problem (II)}$$

$$\text{where } F = \begin{bmatrix} A \\ R \end{bmatrix} \text{ and } f = \begin{bmatrix} b \\ t \end{bmatrix}.$$

Problem (II) may be bounded or unbounded. In case it is unbounded, it can be made bounded by introducing a constraint of the form  $1X \leq M$ ,  $M$  being sufficiently large positive number and  $1$  being sum vector, without losing any of its extreme points. So (II) will always be assumed to be bounded.

We shall also assume that

- (i) the solution set  $S = [X/FX = f, X \geq 0]$  of (II) is regular.
- (ii)  $DX + \beta > 0$  for all  $X \in S$ .

Under these assumptions optimal solution of (II) will occur at an extreme point of  $FX = f, X \geq 0$  (i.e., optimal solution of (II) will be a basic feasible solution) [Swarup, 1965].

The extreme points of  $FX = f, X \geq 0$  will be enumerated till we get a required optimal solution of (I) which is to be an extreme points of  $RX = t, X \geq 0$ .

*Notations:*

$$J = [r_j/r_j \neq 0, \text{ where } r_j \text{ is } j^{\text{th}} \text{ column of } R].$$

$$J(X) = [r_j \in J/x_j \neq 0 \text{ where } X = [x_1, x_2, \dots, x_n] \text{ is a basic feasible solution of (II)}].$$

$$|J(X)| = \text{number of elements in } J(X).$$

$$S_1 = [X/AX = b \text{ and } X \text{ is an extreme point of } RX = t, X \geq 0].$$

$$S_2 = [X/X \text{ is an extreme point of } FX = f, X \geq 0].$$

$$S = [X/FX = f, X \geq 0].$$

$X_i$  = set of  $i^{\text{th}}$  best extreme point solutions of (II)

$$= [X_{i_1}, X_{i_2}, \dots, X_{i_{s_i}}].$$

$$u_i = \frac{CX_{ij} + \alpha}{DX_{ij} + \beta} = \text{value of the objective function at the elements of } X_i.$$

$B_i$  = Set of all the bases corresponding to the elements of  $X_i$ .

$E_i$  = Set of all the bases adjacent to each element of  $B_i$  and yielding value of the objective function less than  $u_i$ .

(Two bases are said to be adjacent to each other if either of them is obtained from the other by changing only one of its vectors [Hadley, w. y., Kirby, Love and Swarup, 1970, and Dantzig, w. y.]).

$$\text{Notice that } E_i \subseteq \bigcup_{j \geq i+1} (B_j).$$

$H_i$  = Set of the bases adjacent to  $I^{\text{st}}, II^{\text{nd}}, \dots, i^{\text{th}}$  best extreme points of (II) leaving  $I^{\text{st}}, II^{\text{nd}}, \dots, i^{\text{th}}$  best extreme points of (II).

$$= \bigcup_{j=1}^i E_j \setminus \bigcup_{j=2}^i B_j.$$

$$\text{Note that } H_i \subseteq \bigcup_{j=1}^i E_j.$$

$$\text{Clearly } H_1 = E_1 \text{ and } \bigcup_{j \geq i} B_j \subseteq H_{i-1}.$$

$\phi$  = Null vector.

*Note:*

(i) The set  $X_i$  of  $i^{\text{th}}$  best extreme points of (II) is adjacent to some element of

$$\bigcup_{j=1}^{i-1} X_j \text{ [Kirby, Love and Swarup, 1972].}$$

(ii)  $X_i \in S_2 \setminus \bigcup_{j=1}^{i-1} X_j$  and is such that

$$u_i \geq \frac{CX + \alpha}{DX + \beta}$$

where  $X$  is any extreme point belonging to  $S_2 \setminus \bigcup_{j=1}^{i-1} X_j$ .

(iii) It is quite obvious that every point of  $S_1$  is a point of  $S$ , converse may not be true. That is  $S_1 \subseteq S$ .

*Theorem 1:*

$S_1 \subseteq S_2$ . That is, every extreme point of  $RX = t, X \geq 0$  which is feasible for  $AX = b$  is also an extreme point of  $FX = f, X \geq 0$  [Kirby, Love and Swarup, 1972].

This theorem shows that the optimal solution of (I), if it exists, is given by some extreme point of (II). As  $S$  is regular, there will exist either of the following two possibilities.

- (L. 1): (II) has a solution and (I) also has a solution and  
 (L. 2): (II) has a solution but (I) has no solution.

Firstly the case (L. 1) will be studied when (I) also has a solution and later the case (L. 2) is taken up when (I) has no solution.

As mentioned earlier, to solve (I) extreme points of (II) are to be enumerated in a certain order and the process is to be terminated when we get the required extreme point of  $RX = t, X \geq 0$ . That is, we will move systematically from one point to another point of  $S_2$ , till we reach the required extreme point which will be in  $S_1$ . So at each stage we have to see whether a point of  $S_2$ , obtained at that stage, is a point of  $S_1$  or not. It is shown in Kirby, Love and Swarup [1972] that a point  $X$  of  $S_2$  will be a point of  $S_1$  if the number of non-null columns of  $R$  corresponding to non-zero basic variables in  $X$  is less than or equal to  $p$  and these columns are linearly independent provided ranks of  $F$  and  $R$  are respectively  $m + p$  and  $p$ .

*Procedure:*

First solve (II) to find  $X_1$  and  $u_1$  [Swarup, 1965].  $X_1 \neq \phi$ . If  $X_1 \cap S_1 \neq \phi$ , then any  $X \in X_1 \cap S_1$  will be the required optimal solution of (I). But if  $X_1 \cap S_1 = \phi$  (i.e., no element of  $X_1$  is an extreme point of  $RX = t, X \geq 0$ ), then an optimal solution of (I) must be an element of  $S_1 \cap (S_2 \setminus X_1)$  since by theorem 1 an optimal solution of (I) must be an element of  $S_1 \cap S_2$ . Then proceed to find the set  $X_2$  for which find  $B_1$  and  $E_1$ . Bases of  $E_1$  which yield the greatest value (say  $u_2$ ) of the objective function, generate the set  $X_2$ . There will be a unique second best extreme point solution of (II) (i.e.,  $X_2$  will have a single element) if there is only one element of  $E_1$  which yields  $u_2$ . If  $X_2 = \phi$ , (I) has no solution. If  $X_2 \neq \phi$  and  $X_2 \cap S_1 \neq \phi$ , then any  $X \in X_2 \cap S_1$  will be an optimal solution of (I). But if  $X_2 \cap S_1 = \phi$ , proceed to find  $X_3$ . To determine  $X_3$ , first find  $B_2, E_2$  and  $H_2$ . Bases of  $H_2$  which yield the greatest value (say  $u_3$ ) of the objective function, generate  $X_3$ . If  $X_3 = \phi$ , (I) has no solution. If  $X_3 \neq \phi$  and  $X_3 \cap S_1 \neq \phi$ , then any  $X \in X_3 \cap S_1$  will be an optimal solution of (I). And if  $X_3 \neq \phi$  and  $X_3 \cap S_1 = \phi$ , the process is repeated by finding  $X_4, X_5, \dots$  till for some  $i$  we get  $X_i \neq \phi$  and  $X_i \cap S_1 \neq \phi$  in which case any  $X \in X_i \cap S_1$  will be an optimal solution of (I).

Observe that for each  $X_i$  obtained,  $X_i \subseteq S_2$  and  $X_i \neq \phi$ . The procedure will converge in a finite number of steps since

- (i)  $S_1 \subseteq S_2$ .
- (ii)  $S_2$  has a finite number of elements.
- (iii) no  $X_i$  is repeated because  $u_i > u_{i+1}$ .
- (iv)  $X_i \cap X_j = \phi$  for all  $i$  and  $j, i \neq j$ .

The method developed above enables us to recognise a property of the optimal solution of (I) which is summarized in the following theorem:

*Theorem 2:*

If  $\hat{X}$  is an optimal solution of (I) and  $X_1 \cap S_1 = \phi$ , then  $\hat{X}$  is adjacent to some element of  $S_3 = S_2 \setminus S_1$  [Kirby, Love and Swarup, 1972].

*Proof:*

According to theorem 1,  $\hat{X} \in S_2$ . Let  $\hat{X}$  belong to the set  $X_N$  of the  $N^{\text{th}}$  best extreme points of (II). Say  $\hat{X}$  is the  $j^{\text{th}}$  member of  $X_N$  i.e., it is same as  $X_{N_j}$ . Therefore,  $X_i \cap S_1 = \phi$  for  $i = 1, 2, \dots, N - 1$ . As  $S_3$  is the set of those extreme points of  $FX = f, X \geq 0$  which are not extreme points of  $RX = t, X \geq 0$ , it follows that  $\bigcup_{i=1}^{N-1} X_i \subseteq S_3$ . As  $X_{N_j}$  is adjacent to some element of  $\bigcup_{i=1}^{N-1} X_i$ , it follows that  $X_{N_j}$  is adjacent to an element of  $S_3$ .

This theorem shows that the method described above does not require that all the extreme points of  $S_2$  should be examined. This is what is done in the algorithm. We examine only elements of  $\bigcup_{i=1}^{N-1} X_i$  starting from  $X_1$  (assuming that  $X_{N_j}$  is the optimal solution of (I)) until an  $N^{\text{th}}$  best extreme solution is found which is feasible for (I). Thus points of  $S_3$  are investigated one by one till we reach a point in  $S_1$  where the procedure is terminated.

We have still to study the case (L. 2) where (II) has a solution but (I) has no solution. That is,  $S_2 \cap S_1 = \phi$ . In this case it will be necessary to examine all the extreme points of (II). As for each  $i, X_i \cap S_1 = \phi$ , it seems that the procedure will never stop. But this will not be the case because  $S_2$  has a finite number of elements and further  $u_i > u_{i+1}$ . So after a finite number of steps, say  $N$ , it will be impossible to get  $(N + 1)^{\text{th}}$  best extreme point of (II).

Thus if a stage is reached when no further best extreme point solution is possible, then (I) has no solution. The fact that (II) has no further best extreme point solution at some stage (say  $N^{\text{th}}$  stage) is indicated by the fact that at that stage  $H_N = \phi$ .

*Statement of the Algorithm:*

The whole procedure is incorporated in the following algorithm.

*Algorithm:*

*Step 1:* Solve (II). If (II) has an unbounded solution, go to step 5. If (II) has a bounded solution, find  $X_r, B_r$  (starting from  $r = 1$ ) and go to step 2.

*Step 2:* Determine whether  $X_r \cap S_1 \neq \phi$ . If  $X_r \cap S_1 \neq \phi$ , terminate the process. In this case any  $X \in X_r \cap S_1$  will be an optimal solution of (I) yielding value  $U_r$ . If  $X_r \cap S_1 = \phi$ , go to step 3.

*Step 3:* Find  $E_r$  and  $H_r$  (starting from  $r = 1$ ). If  $H_r = \phi$ , then (I) has no solution and the procedure is terminated. If  $H_r \neq \phi$ , then go to step 4.

*Step 4:* Find  $U_{r+1}$  and determine the sets  $B_{r+1}$  and  $X_{r+1}$  and go to step 2.

*Step 5:* Rewrite (II) to include the constraint  $1X \leq M$ , where  $M$  is a sufficiently large positive number and  $1$  is a sum vector, and return to step 1.

*Remarks:*

1. However, if degeneracy of (II) is to be considered, it can be dealt with in a manner similar to the one adopted in ordinary linear fractional functional programming problem (degenerate) or linear programming problem (degenerate) [Hadley, w. y.].
2. Note that when an extreme point solution, say  $X$ , of (II) is reached which is not an optimal solution of (I), then we examine only those of its adjacent extreme points which yield lesser value (less than the value yielded by  $X$ ) of the objective function. Such adjacent extreme points are obtained by entering those vectors in the basis of the given extreme point (i.e.,  $X$ ) for which  $\Delta_j$ 's are negative [Swarup, 1965]. (In numerical example,  $\Delta_j$ 's are defined on page 8.)

*Example:*

$$\begin{aligned} \text{Max } Z &= \frac{2x_1 + x_2}{4x_1 + x_2 + 1} \\ \text{subject to } & -2x_1 + x_2 \leq 1 \\ & 2x_1 + 5x_2 \leq 23 \\ & 2x_1 + x_2 \leq 15 \\ \text{and } (x_1, x_2) & \text{ is an extreme point of} \\ & -3x_1 + 2x_2 \leq 4 \\ & x_1 + 4x_2 \leq 22 \\ & 5x_1 + 4x_2 \leq 46 \\ & x_1 - 2x_2 \leq 5 \\ & x_1, x_2 \geq 0. \end{aligned}$$

After adding slack variables the problem (I) of theory assumes the following form:

$$\begin{aligned} \text{Max } Z &= \frac{2x_1 + x_2}{4x_1 + x_2 + 1} && | && && | \\ \text{subject to } & -2x_1 + x_2 + x_3 = 1 && | && && | \\ & 2x_1 + 5x_2 + x_4 = 23 && | \equiv & A & X = b & | \\ & 2x_1 + x_2 + x_5 = 15 && | && && | \\ \text{and } (x_1, x_2, \dots, x_9) & \text{ is an extreme point of} && | && && | \quad \dots \text{ Problem (I)} \\ & -3x_1 + 2x_2 + x_6 = 4 && | && && | \\ & x_1 + 4x_2 + x_7 = 22 && | \equiv & R & X = t & | \\ & 5x_1 + 4x_2 + x_8 = 46 && | && && | \\ & x_1 - 2x_2 + x_9 = 5 && | && && | \\ & x_1, x_2, \dots, x_9 \geq 0 && | && && | \end{aligned}$$

$A$  is  $3 \times 9$ ,  $X$  is  $9 \times 1$ ,  $R$  is  $4 \times 9$ . Here  $p$  is  $= 4$ . Problem (II) of theory takes the following form

$$\begin{array}{l}
 \text{Max } Z = \frac{2x_1 + x_2}{4x_1 + x_2 + 1} \\
 \text{subject to } \begin{array}{l}
 -2x_1 + x_2 + x_3 = 1 \\
 2x_1 + 5x_2 + x_4 = 23 \\
 2x_1 + x_2 + x_5 = 15 \\
 -3x_1 + 2x_2 + x_6 = 4 \\
 x_1 + 4x_2 + x_7 = 22 \\
 5x_1 + 4x_2 + x_8 = 46 \\
 x_1 - 2x_2 + x_9 = 5 \\
 x_1, x_2, \dots, x_9 \geq 0
 \end{array}
 \end{array} \equiv |FX = f| \quad \dots \text{ Problem (II)}$$

Step 1:  $X_1 = [X_{11} = (3/2, 4, 0, 0, 8, \frac{1}{2}, 9/2, 45/2, 23/2)]$   
 $u_1 = 7/11$   
 $B_1 = [B_{11} = (f_2, f_1, f_5, f_6, f_7, f_8, f_9)]$

where  $f_1, f_2, \dots, f_9$  are columns of  $F$ .

$$\begin{array}{l}
 J(X_{11}) = r_1, r_2, r_6, r_7, r_8, r_9 \\
 |J(X_{11})| = 6 > 4 \\
 \therefore X_1 \cap S_1 = \phi.
 \end{array}$$

Tableau for  $X_{11}$  is:

				$c_j \rightarrow$	2	1	0	0	0	0	0	0	0
				$d_j \rightarrow$	4	1	0	0	0	0	0	0	0
$D_{B_{11}}$	$C_{B_{11}}$	Vectors in basis	$X_{B_{11}}$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$f_7$	$f_8$	$f_9$	
1	1	$f_2$	4	0	1	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0	0	
4	2	$f_1$	$\frac{3}{2}$	1	0	$-\frac{5}{12}$	$\frac{1}{12}$	0	0	0	0	0	
0	0	$f_5$	8	0	0	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	0	0	0	
0	0	$f_6$	$\frac{1}{2}$	0	0	$-\frac{19}{12}$	$-\frac{1}{12}$	0	1	0	0	0	
0	0	$f_7$	$\frac{9}{2}$	0	0	$-\frac{1}{4}$	$-\frac{3}{4}$	0	0	1	0	0	
0	0	$f_8$	$\frac{45}{2}$	0	0	$\frac{17}{12}$	$-\frac{13}{12}$	0	0	0	1	0	
0	0	$f_9$	$\frac{23}{2}$	0	0	$\frac{3}{4}$	$\frac{1}{4}$	0	0	0	0	1	
$z^{(2)} = 11, z^{(1)} = 7, c_j - z_j^{(1)} \rightarrow$				$z = \frac{7}{11}$	0	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	0	0	
$d_j - z_j^{(2)} \rightarrow$					0	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	0	0	
$\Delta_j \rightarrow$					0	0	$-\frac{19}{6}$	$-\frac{1}{6}$	0	0	0	0	

where  $z^{(1)} = C_{B_{11}} X_{B_{11}}$   
 $z^{(2)} D_{B_{11}} X_{B_{11}} + 1.$

$C_{B_{11}}$  and  $D_{B_{11}}$  are the vectors having their components as the coefficients associated with the basic variables in the numerator and denominator of the objective function.

$$\begin{array}{l}
 z_j^{(1)} = C_{B_{11}} V_j, \quad z_j^{(2)} = D_{B_{11}} V_j \\
 V_j = B_{11}^{-1} f_j
 \end{array}$$

$$Z = \frac{Z^{(1)}}{Z^{(2)}}$$

$$\Delta_j = Z^{(2)} (c_j - z_j^{(1)}) - z^{(1)}(d_j - z_j^{(2)}).$$

$C_j$  is the  $j^{\text{th}}$  element of  $C$  and  $d_j$  is the  $j^{\text{th}}$  element of  $D$ .

Step 2:  $E_1 = [E_{11} = (f_2, f_1, f_3, f_6, f_7, f_8, f_9), E_{12} = (f_2, f_4, f_5, f_6, f_7, f_8, f_9)] \frac{15}{29}$  is the value of the objective function yielded by  $(f_2, f_1, f_3, f_6, f_7, f_8, f_9), \frac{1}{2}$  is the value of the objective function yield by  $(f_2, f_4, f_5, f_6, f_7, f_8, f_9)$

$$u_2 = \frac{15}{29}$$

$$B_2 = [B_{21} = (f_2, f_1, f_3, f_6, f_7, f_8, f_9)]$$

$$X_2 = [X_{21} = (\frac{13}{2}, 2, 12, 0, 0, \frac{39}{2}, \frac{15}{2}, \frac{11}{2}, \frac{5}{2})]$$

$$J(X_{21}) = r_1, r_2, r_6, r_7, r_8, r_9$$

$$|J(X_{21})| = 6 > 4$$

$\therefore X_{21}$  is not an extreme point of  $RX = t, X \geq 0$

$\therefore X_2 \cap S_1 = \phi$ .

Step 3:  $E_2 = [E_{21} = (f_2, f_1, f_3, f_6, f_7, f_8, f_4)]$

$$H_2 = E_1 \cup E_2 \setminus B_2 = [(f_2, f_4, f_5, f_6, f_7, f_8, f_9), (f_2, f_1, f_3, f_6, f_7, f_8, f_4)]$$

$\frac{1}{2}$  is the value of the objective function yielded by  $(f_2, f_4, f_5, f_6, f_7, f_8, f_9)$

$\frac{1}{2}$  is the value of the objective function yielded by  $(f_2, f_1, f_3, f_6, f_7, f_8, f_4)$

$$B_3 = [B_{31} = (f_2, f_4, f_5, f_6, f_7, f_8, f_9), B_{32} = (f_2, f_1, f_3, f_6, f_7, f_8, f_4)]$$

$$X_3 = [X_{31} = (0, 1, 0, 18, 14, 2, 18, 42, 7), X_{32} = (7, 1, 14, 4, 0, 23, 11, 7, 0)]$$

$$J(X_{31}) = r_2, r_6, r_7, r_8, r_9$$

$$|J(X_{31})| = 5 > 4 \therefore X_{31} \text{ is not an extreme point of } RX = t, X \geq 0.$$

$$J(X_{32}) = r_1, r_2, r_6, r_7, r_8$$

$$|J(X_{32})| = 5 > 4 \therefore X_{32} \text{ is not an extreme point of } RX = t, X \geq 0.$$

$\therefore X_3 \cap S_1 = \phi$ .

Step 4:  $E_3 = [E_{31} = (f_3, f_4, f_5, f_6, f_7, f_8, f_9), E_{32} = (f_5, f_1, f_3, f_6, f_7, f_8, f_4)]$

$$H_3 = E_1 \cup E_2 \cup E_3 \setminus B_2 \cup B_3$$

$$= [(f_3, f_4, f_5, f_6, f_7, f_8, f_9), (f_5, f_1, f_3, f_6, f_7, f_8, f_4)]$$

0 is the value of the objective function yielded by  $(f_3, f_4, f_5, f_6, f_7, f_8, f_9)$ .

$\frac{10}{21}$  is the value of objective function yield by  $(f_5, f_1, f_3, f_6, f_7, f_8, f_4)$ .

$$\therefore u_4 = \frac{10}{21}$$

$$B_4 = [B_{41} = (f_5, f_1, f_3, f_6, f_7, f_8, f_4)]$$

$$X_4 = [X_{41} = (5, 0, 11, 13, 5, 19, 17, 21, 0)]$$

$$J(X_{41}) = r_1, r_6, r_7, r_8$$

$$|J(X_{41})| = 4$$



Also  $r_1, r_6, r_7, r_8$  are linearly independent

$\therefore X_{41}$  is an extreme point  $RX = t, X \geq 0$ .

$\therefore X_4 \cap S_1 \neq \phi$ .

Therefore, optimal solution of (I) is:

$$X_{41} = (5, 0, 11, 13, 5, 19, 17, 21, 0) \text{ yielding value } \frac{10}{21}.$$

Therefore, optimal solution of the original problem is

$$x_1 = 5, \quad x_2 = 0$$

and the optimal value is  $\frac{10}{21}$ .

## References

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