Top

Volume 12, Number 2, 301-330
December 2004

REPRINT

Sonia and M.C. Puri

Two Level Hierarchical Time Minimizing Transportation Problem

Published by
Sociedad de Estadística e Investigación Operativa
Madrid, Spain

Top

Volume 12, Number 2
December 2004

Published by
Sociedad de Estadística e Investigación Operativa
Madrid, Spain

Sociedad de Estadística e Investigación Operativa
Top (2004) Vol. 12, No. 2, pp. 301–330

Two Level Hierarchical Time Minimizing Transportation Problem

Sonia and Munish C. Puri
Department of Mathematics
Indian Institute of Technology, Delhi,
Hauz-Khas, New Delhi-110016, India
E-mail: sonia_iit@rediffmail.com mcpuri@maths.iitd.ernet.in

Abstract

A two level hierarchical balanced time minimizing transportation problem is considered in this paper. The whole set of source-destination links consists of two disjoint partitions namely Level-I links and Level-II links. Some quantity of a homogeneous product is first shipped from sources to destinations by Level-I decision maker using only Level-I links, and on its completion the Level-II decision maker transports the remaining quantity of the product in an optimal fashion using only Level-II links. Transportation is assumed to be done in parallel in both the levels. The aim is to find that feasible solution for Level-I decision maker corresponding to which the optimal feasible solution for Level-II decision maker is such that the sum of shipment times in Level-I and Level-II is the least. To obtain the global optimal feasible solution of this non-convex optimization problem, related balanced time minimizing transportation problems are defined. Based upon the optimal feasible solutions of these related problems, standard cost minimizing transportation problems are constructed whose optimal feasible solutions provide various pairs for shipment times for Level-I and Level-II decision makers. The best out of these pairs is finally selected. Being dependent upon solutions of a finite number of balanced time minimizing and cost minimizing transportation problems, the proposed algorithm is a polynomial bound algorithm. The developed algorithm has been implemented and tested on a variety of test problems and performance is found to be quite encouraging.

Key Words: Global optimization, concave minimization problem, time minimizing transportation problem, hierarchical optimization.

AMS subject classification: 90C27, 90C26, 90C08, 90C90.

We are thankful to Prof. S. N. Kabadi, University of New Brunswick-Fredericton, Canada, who initiated us to this type of problems. We are also thankful to Mr. Avneet Bansal, who helped us in the implementation of the algorithm.

Manuscript received: April 2003. Final version accepted: April 2004.

1 Introduction

The problem discussed in this paper belongs to the class of concave minimization problem (CMP). Locatelli and Thoai (Locatelli and Thoai(2000)) discussed a simplicial branch and bound algorithm for a general concave minimization problem. In general, concave minimization problems are hard problems except their special cases like concave minimization flow problem, product transportation problem, freight transportation problem and time minimizing transportation problem etc. A very brief description of these is given below.

The general minimum concave cost flow problem (MCCFP) is known to be NP-hard. Horst and Thoai (Horst and Thoai (1988)) proposed a finite branch-and-bound method for solving the general minimum concave cost flow problem in which the branching operation involves suitable integral rectangular partitions and bounding operation calls for solving a minimum linear cost flow problem on subnetworks.

An important class of concave minimization problem namely the minimum cost production transportation problem (PTP) with concave production costs has been solved using branch-and-bound algorithm of Falk Soland's type (Kuno and Utsunomia (2000)). To accelerate the convergence of the algorithm, they reinforce the bounding operation using a Langrangian relaxation, which is a concave minimization but yields a tighter bound than usual linear programming relaxation. Tuy et al. (Tuy et al. (1996)) gave a polynomial algorithm for production transportation problem involving an arbitrary fixed number of factories with concave production cost. A primal-dual algorithm for solving a class of production transportation problem is proposed by Kuno and Utsunomia (Kuno and Utsunomia (1996)).

Another class of concave minimization problem is freight transportation problem (FTP) (Klincewicz (1990)). In freight transportation problem sources can ship in bulk to one or more intermediate terminals (called consolidation terminals) and at these terminals, shipment from many sources can be consolidated for eventual shipment to the various destinations. Shipment costs are piecewise linear concave functions of the volume shipped and shipping via a consolidation terminal incurs a linear inventory holding cost. Minimum cost solution of direct or indirect shipments were desired.

A very important class of concave minimization problem which has enormous applications is the time minimizing transportation problem (TMTP).

It usually arises in connection with transportation of perishable commodities that have to be distributed as quickly as possible. Many authors (Hammer (1969), Garfinkel and Rao (1971), Szwarc (1971), Ahuja et al. (1994)) have contributed to this area of concave minimization problem. The problem discussed in this paper is also related to time minimizing transportation problem. A brief description of this (TMTP) is given below.

Consider a transportation problem defined by a set I of m sources and a set J of n destinations. In a balanced transportation problem total availability, say $\sum_{i\in I} a_i$, matches the total demand $\sum_{j\in J} b_j$, where a_i is the availability of a homogeneous product at the source $i \in I$ and b_j is the requirement of the same at the destination $j \in J$. For each source-destination link $(i,j) \in I \times J, x_{ij} \geq 0$ denotes the shipment from the source i to the destination j and $\mathsf{t}_{ij}(x_{ij})$ denotes the associated shipment time defined as:

$$\begin{aligned} \mathsf{t}_{ij}(x_{ij}) &= t_{ij}(\geq 0), \quad \text{if} \quad x_{ij} > 0 \\ &= 0, \qquad\qquad \text{otherwise.} \end{aligned}$$

Clearly each $\mathsf{t}_{ij}(x_{ij})$ is a concave function. When the transportation from the sources to the destinations is done in parallel, then the overall shipment time $T(X)$ for a feasible schedule X is defined as: $T(X) = \max_{I\times J}[\mathsf{t}_{ij}(x_{ij})]$.

Bansal and Puri (Bansal and Puri (1980)) proved that $T(X)$ is a concave function. The time minimizing transportation problem is modelled as:

$$\min_{X\in S} T(X)$$

where S is a transportation polytope defined as:

$$S = \left\{ X = (x_{ij}) \in R^{mn} : \begin{array}{rll} \sum_{j\in J} x_{ij} & = a_i & i \in I \\ \sum_{i\in I} x_{ij} & = b_j, & j \in J \\ x_{ij} & \geq 0 & \forall\ (i,j) \in I \times J \end{array} \right\}$$

Special combinatorial structure of S significantly reduces the complexity of the problem. Almost all methods for solving time minimizing transportation problem involve solving an ordinary cost minimizing transportation problem (CMTP). As cost minimizing transportation problem is known to be solvable in strongly polynomial time (Tardos (1985), Tardos (1986)), it follows that time minimizing transportation problem is also solvable in strongly polynomial time. Solvability in strongly polynomial time means that there exists an algorithm which solves the problem in a number of steps that is bounded by a polynomial function of m and n only. The best strongly polynomial running time to date for the cost minimizing transportation problem is $O\big(\overline{m+n}\ \log mn\ (\overline{m+n} + mn \log mn)\big)$. This bound is achieved by an application of a minimum cost flow algorithm of Orlin (Orlin (1988)) to cost minimizing transportation problem. A slight improvement over this bound is obtained by Kleinschmidt and Schannath (Kleinschmidt and Schannath (1995)). They proposed an algorithm for cost minimizing transportation problem which runs in time proportional to $m \log m(k + n \log n)$ where, k is the number of feasible links. Sharma and Sharma (Sharma and Sharma (2000)) have proposed a computationally attractive $O(c\bar{n}^2)$ dual based heuristic procedure for solving a cost minimizing transportation problem where c is a constant and $\bar{n} = |I| + |J|$.

In the present paper, a two-level hierarchical time minimizing transportation problem is considered in which all the source-destination links are grouped in two categories viz. Level-I links and Level-II links. In Level-I, the leader can use only Level-I links for shipment of goods from the sources to the destinations. On the completion of shipment in Level-I, the follower uses Level-II links optimally to transport the left-over quantity. Clearly solution space for the follower in Level-II is dependent upon the feasible shipment schedule used by the leader in the transportation of goods in Level-I. Transportation is assumed to be done in parallel in both the levels. Since the transportation time for the leader in the Level-I is a concave function and that for the follower in Level-II is also a concave function, it follows that the overall transportation time for the two level hierarchical time minimizing transportation problem would be a concave function. Aim is to find that feasible shipment schedule for the leader in Level-I so that the corresponding optimal shipment schedule for the follower in Level-II is such that the overall shipment time for the two level hierarchical time minimizing transportation problem is the least. Clearly the two level hierarchical time minimizing transportation problem is a concave minimization problem. The main difficulty of two level hierarchical time minimizing transportation problem comes, of course, from the nonconvexity of the objective function. On the other hand, a nice structure underlines the two level hierarchical time minimizing transportation problem. However, due to concavity of the objective function, it turns out that the search for an optimal solution of two level hierarchical time minimizing transportation problem can just be restricted to the set of the vertices of the transportation polytope. Exploiting the relation of the two level hierarchical time minimizing transportation problem with the standard time minimizing transportation problem, a polynomial algorithm is proposed.

To solve the above mentioned two level hierarchical time minimizing transportation problem, first a related standard time minimizing transportation problem is solved. Based upon its solution a cost minimizing transportation problem is constructed whose optimal basic feasible solution (OBFS) yields the first feasible solution of the two level hierarchical time minimizing transportation problem. A time minimizing transportation problem with respect to Level-II shipment time is defined. Based upon its optimal feasible solution (OFS) a cost minimizing transportation problem is constructed whose optimal basic feasible solution yields the second feasible solution of the two level hierarchical time minimizing transportation problem. This process of defining a time minimizing transportation problem with respect to Level-II shipment time and constructing cost minimizing transportation problem based upon its optimal feasible solution is continued to generate various feasible solutions of the two level hierarchical time minimizing transportation problem. These solutions yield pairs $\Big(T_{L_1}(\cdot), T_{L_2}(\cdot) : T_{L_1}(\cdot) > T_{L_2}(\cdot)\Big)$ where $T_{L_1}(\cdot)$ and $T_{L_2}(\cdot)$ denote Level-I and Level-II shipment times respectively. Later pairs of the form $\Big(T_{L_1}(\cdot), T_{L_2}(\cdot) : T_{L_1}(\cdot) < T_{L_2}(\cdot)\Big)$ are also obtained by defining the time minimizing transportation problems with respect to various successive values of $T_{L_1}(\cdot)$. Based upon their solutions, the cost minimizing transportation problems are constructed whose solutions provide more feasible solutions of the two level hierarchical time minimizing transportation problem. These solutions yield the pairs $\Big(T_{L_1}(\cdot), T_{L_2}(\cdot) : T_{L_1}(\cdot) < T_{L_2}(\cdot)\Big)$. Finally a feasible solution of the two level hierarchical time minimizing transportation problem corresponding to the minimum value of $T_{L_1}(\cdot) + T_{L_2}(\cdot)$ provides a global optimal feasible solution of the problem.

The plan of the paper is as follows: In the next section on theoretical development, mathematical model of the two level hierarchical time minimizing transportation problem is given and various results are established. Based upon these results, an algorithm is proposed in the section 3 that solves the two level hierarchical time minimizing transportation problem in a polynomial time. Numerical illustration is given in the section 4 and section 5 contains concluding remarks.

2 Theoretical Development

The set of source-destination links in $I \times J$ is partitioned into two non-empty disjoint sets L_1 and L_2 such that L_1 consists of links $(i, j) \in I \times J$ to be used in shipment pertaining to Level-I. Therefore, $L_2 = I \times J \backslash L_1$ represents the set of Level-II links. It is assumed that transportation is done in parallel in both the levels. The Level-I transportation problem in which the transportation is done using only Level-I links is mathematically defined as:

$$\min_{Y \in S_{L_1}} \left[\max_{L_1} \left(\mathsf{t}_{ij}(y_{ij}) \right) \right]$$

where,

$$S_{L_1} : \begin{cases} \sum\limits_{j:(i,j) \in L_1} y_{ij} & \leq & a_i, \ i \in I \\ \sum\limits_{i:(i,j) \in L_1} y_{ij} & \leq & b_j, \ j \in J \\ y_{ij} & \geq & 0, \forall \ (i,j) \in L_1 \end{cases}$$

For a feasible solution Y of Level-I transportation problem, the Level-II transportation problem is defined as:

$$\min_{Z \in S_{L_1}(Y)} \left[\max_{L_2} \left(\mathsf{t}_{ij}(z_{ij}) \right) \right]$$

where,

$$S_{L_1}(Y) : \begin{cases} \sum\limits_{j:(i,j) \in L_2} z_{ij} & = a_i - a'_i, & i \in I \\ \sum\limits_{i:(i,j) \in L_2} z_{ij} & = b_j - b'_j, & j \in J \\ z_{ij} & \geq 0, & \forall \ (i,j) \in L_2 \end{cases}$$

and $a'_i = \sum\limits_{j:(i,j) \in L_1} y_{ij}$ is the quantity shipped from i^{th} source and $b'_j = \sum\limits_{i:(i,j) \in L_1} y_{ij}$ is the quantity shipped to j^{th} destination in Level-I transportation problem.

Thus the two-level hierarchical time minimizing transportation problem is defined as:

$$\min_{Y\in S_{L_1}}\Big[\max_{L_1}\big(\mathsf{t}_{ij}(y_{ij})\big)+\min_{Z\in S_{L_1}(Y)}\big\{\max_{L_2}\big(\mathsf{t}_{ij}(z_{ij})\big)\big\}\Big] \tag{HTP}$$

Clearly a feasible solution of Level-I transportation problem, say $Y \in S_{L_1}$, along with the corresponding optimal feasible solution of Level-II transportation problem constitutes a feasible solution of the two level hierarchical time minimizing transportation problem (HTP). Clearly (HTP) is closely related to the following balanced time minimizing transportation problem defined as:

$$\min_{X\in S}\Big[\max_{I\times J}\big(\mathsf{t}_{ij}(x_{ij})\big)\Big] \tag{TP}$$

where,

$$S:\begin{cases}\displaystyle\sum_{j\in J} x_{ij} &= a_i, \quad i\in I\\ \displaystyle\sum_{i\in I} x_{ij} &= b_j, \quad j\in J\\ \qquad x_{ij} &\geq 0, \quad \forall\ (i,j)\in I\times J\end{cases}$$

Suppose $L_1^k = \{(i,j) \in L_1 : t_{ij} = T_{L_1}^k\},\ k = 0,1,\dots,l_1$ be the pairwise disjoint partitions of the Level-I source-destination links where, $T_{L_1}^j > T_{L_1}^{j+1},\ j = 1,2,\dots,(l_1-1)$.

Similarly $L_2^k = \{(i,j) \in L_2 : t_{ij} = T_{L_2}^k\},\ k = 0,1,\dots,l_2$ denote pairwise distinct partitions of the Level-II source-destination links where, $T_{L_2}^j > T_{L_2}^{j+1},\ j = 1,2,\dots,(l_2-1)$.

Suppose X is an optimal basic feasible solution of the time minimizing transportation problem (TP). Let Level-I and Level-II shipment times for this feasible schedule X be respectively $T_{L_1}^u$ and $T_{L_2}^v$ where, $T_{L_1}^u \in \{T_{L_1}^k, k = 1,2,\dots,l_1\}$ and $T_{L_2}^v \in \{T_{L_2}^k, k = 1,2,\dots,l_2\}$. That is, $T_{L_1}(X) = \max\limits_{L_1}\{\mathsf{t}_{ij}(x_{ij})\} = T_{L_1}^u$, and $T_{L_2}(X) = \max\limits_{L_2}\{\mathsf{t}_{ij}(x_{ij})\} = T_{L_2}^v$.

Without loss of generality it may be assumed that $T_{L_1}^u \geq T_{L_2}^v$. This means overall minimum shipment time in the time minimizing transportation problem (TP) is $T_{L_1}^u = \max\limits_{I\times J}\{\mathsf{t}_{ij}(x_{ij})\}$.

The optimal basic feasible solution of the cost minimizing transportation problem $(CP_{L_2}^v)$ defined below will provide the minimum Level-II shipment time (say $T_{L_2}^{v+q_0} \leq T_{L_2}^{v},\ q_0 \geq 0$) corresponding to the Level-I shipment time $T_{L_1}^{u}$. (Ref. Theorem 2.2)

$$\min_{X \in S} \sum_{I \times J} c_{ij} x_{ij} \qquad (CP_{L_2}^{v})$$

where,

$$\begin{aligned} c_{ij} &= M, && (i,j) \in L_1 : t_{ij} > T_{L_1}^{u}, \text{ or } (i,j) \in L_2 : t_{ij} > T_{L_2}^{v} \\ &= 0, && (i,j) \in L_1 : t_{ij} \leq T_{L_1}^{u} \\ &= \lambda_{v+r}, && (i,j) \in L_2 : t_{ij} = T_{L_2}^{v+r}, r = 0, 1, \ldots, (l_2 - v) \end{aligned}$$

λ_j's being positive integers such that $\lambda_j >> \lambda_{j+1}\ \forall\ j$.

For specification of λ_j's one may refer to Mazzola (Mazzola (1983)) and Sherali (Sherali (1982)). The first pair of the Level-I and Level-II shipment times thus generated is $\left(T_{L_1}^{u-p_0}, T_{L_2}^{v+q_0} : T_{L_1}^{u-p_0} \geq T_{L_2}^{v+q_0}\right),\ p_0 = 0$.

Suppose (T^{u-p_0}, T^{v+q_0}), $(T_{L_1}^{u-p_1}, T_{L_2}^{v+q_1})$, $\ldots$, $(T_{L_1}^{u-p_{k-1}}, T_{L_2}^{v+q_{k-1}})$ are the pairs of Level-I and Level-II shipment times obtained so far where, $p_1, p_2, \ldots, p_{k-1}$ and $q_1, q_2, \ldots, q_{k-1}$ are positive integers such that $p_{j+1} > p_j$ and $q_{j+1} > q_j$ for all $j = 0, 1, 2, \ldots, k-2$. These pairs are such that $T_{L_2}^{v+q_j} (j = 0, 1, 2, \ldots, k-1)$ is the minimum Level-II shipment time corresponding to the time $T_{L_1}^{u-p_j}$ of Level-I shipment and $T_{L_1}^{u-p_j}$ is the minimum Level-I shipment time corresponding to the time $T_{L_2}^{v+q_j}$ of Level-II shipment (Ref. Theorem 2.2).

To find the next higher value of the Level-I shipment time (say $T_{L_1}^{u-p_k} > T_{L_1}^{u-p_{k-1}}$)(Ref. Theorem 2.1), the following time minimizing transportation problem (call it $(TP_{L_2}^{v+q_{k-1}})$) is studied

$$\min_{X \in S} \left\{ \max_{I \times J} \left(\mathrm{t}'_{ij}(x_{ij}) \right) \right\} \qquad (TP_{L_2}^{v+q_{k-1}})$$

where,

$$\begin{aligned} t'_{ij} &= M(>> 0), && (i,j) \in L_2 : t_{ij} \geq T_{L_2}^{v+q_{k-1}} \\ &= t_{ij}, && \text{otherwise} \end{aligned}$$

A feasible solution of this problem is called an **M-feasible solution** if

$x_{ij} = 0 \ \forall \ (i,j) \in I \times J$ for which $t'_{ij} = M$.

Theorem 2.1. *Let $X_{L_2}^{v+q_{k-1}} = (x_{L_{2ij}}^{v+q_{k-1}})$ be an optimal basic M-feasible solution of the time minimizing transportation problem $(TP_{L_2}^{v+q_{k-1}})$. Then, the Level-I shipment time at this optimal solution is greater than $T_{L_1}^{u-p_{k-1}}$.*

Proof. Since optimal basic feasible solution of the problem $(TP_{L_2}^{v+q_{k-1}})$ is M-feasible, it follows from the construction of this problem that $T_{L_2}(X_{L_2}^{v+q_{k-1}}) < T_{L_2}^{v+q_{k-1}}$.

Let $T_{L_2}(X_{L_2}^{v+q_{k-1}}) = T_{L_2}^{v+q_k^0}(< T_{L_2}^{v+q_{k-1}})$ where, q_k^0 is positive integer greater than q_{k-1}.

As $T_{L_2}^{v+q_{k-1}}$ is the minimum Level-II shipment time corresponding to time $T_{L_1}^{u-p_{k-1}}$ of Level-I shipment and as $T_{L_2}(X_{L_2}^{v+q_{k-1}}) < T_{L_2}^{v+q_{k-1}}$, it follows that $T_{L_1}(X_{L_2}^{v+q_{k-1}}) > T_{L_1}^{u-p_{k-1}}$. □

As $T_{L_1}(X_{L_2}^{v+q_{k-1}}) > T_{L_1}^{u-p_{k-1}}$, let $T_{L_1}(X_{L_2}^{v+q_{k-1}}) = T_{L_1}^{u-p_k}$ where, $p_k(> p_{k-1})$ is a positive integer.

To obtain the minimum value of the Level-II shipment time corresponding to the time $T_{L_1}^{u-p_k}$ of Level-I shipment (Ref. Theorem 2.2), the following cost minimizing transportation problem is defined:

$$\min_{X \in S} \sum_{I \times J} c_{ij} x_{ij} \qquad (CP_{L_2}^{v+q_k^0})$$

where,

$$\begin{aligned} c_{ij} &= M, && (i,j) \in L_1 : t_{ij} > T_{L_1}^{u-p_k}, \text{ or } (i,j) \in L_2 : t_{ij} > T_{L_2}^{v+q_k^0} \\ &= 0, && (i,j) \in L_1 : t_{ij} \leq T_{L_1}^{u-p_k} \\ &= \lambda_{v+q_k^0+r}, && (i,j) \in L_2 : t_{ij} = T_{L_2}^{v+q_k^0+r}; r = 0, 1, \ldots, (l_2 - (v + q_k^0)) \end{aligned}$$

and λ_j's are positive integers such that $\lambda_j >> \lambda_{j+1} \ \forall \ j$.

Clearly optimal basic M-feasible solution of $(TP_{L_2}^{v+q_{k-1}})$ is a feasible solution of this cost minimizing transportation problem $(CP_{L_2}^{v+q_k^0})$ yielding non-zero value of its objective function. Therefore, the optimal value of the objective function in $(CP_{L_2}^{v+q_k^0})$ is non-negative. The theorem (Theorem 2.2) that follows establishes that the minimum value of the Level-II shipment corresponding to the time $T_{L_1}^{u-p_k}$ of the Level-I shipment is yielded by an optimal basic feasible solution of this problem.

Theorem 2.2. *Let $X_{L_2}^{v+q_k^0}$ be an optimal basic feasible solution of the cost minimizing transportation problem $(CP_{L_2}^{v+q_k^0})$. Then, $T_{L_2}(X_{L_2}^{v+q_k^0})$ is the minimum Level-II shipment time corresponding to the time $T_{L_1}^{u-p_k}$ of the Level-I shipment. Also $T_{L_1}^{u-p_k}$ is the minimum Level-I shipment time corresponding to the time $T_{L_2}(X_{L_2}^{v+q_k^0})$ of the Level-II shipment.*

Proof. As mentioned above, the optimal value of the objective function of the cost minimizing transportation problem $(CP_{L_2}^{v+q_k^0})$ is non-negative.

Case-I. The optimal value of the objective function of the problem $(CP_{L_2}^{v+q_k^0})$ is zero.

In this case $T_{L_2}(X_{L_2}^{v+q_k^0}) = 0$.

$T_{L_1}(X_{L_2}^{v+q_k^0}) \not< T_{L_1}^{u-p_k}$ because for Level-I shipment time smaller than $T_{L_1}^{u-p_k}$ the corresponding minimum time for Level-II shipment is not zero. Therefore, $T_{L_1}(X_{L_2}^{v+q_k^0}) = T_{L_1}^{u-p_k}$. The value of the objective function of the problem (HTP) is $(T_{L_1}^{u-p_k} + 0)$.

Case-II. The optimal value of the objective function of the problem $(CP_{L_2}^{v+q_k^0})$ is positive.

In this case $T_{L_2}(X_{L_2}^{v+q_k^0}) \leq T_{L_2}^{v+q_k^0}$.

Let $T_{L_2}(X_{L_2}^{v+q_k^0}) = T_{L_2}^{v+q_k}(\leq T_{L_2}^{v+q_k^0})$ where, q_k is a positive integer not less than the positive integer q_k^0. Let $q_k = q_k^0 + j$ for some $j \geq 0$. It is claimed that $T_{L_2}^{v+q_k}$ is the minimum Level-II shipment time corresponding to the time $T_{L_1}^{u-p_k}$ of Level-I shipment.

Let $T_{L_2}^{v+q_k}$ be not the minimum Level-II shipment time corresponding to the time $T_{L_1}^{u-p_k}$ of Level-I shipment.

This implies that there exists a solution $Y \in S$ such that $T_{L_2}(Y) = T_{L_2}^{v+q_k+s}(< T_{L_2}^{v+q_k})$, $s \geq 1$ and $T_{L_1}(Y) = T_{L_1}^{u-p_k}$.

Hence $T_{L_2}(Y) = T_{L_2}^{v+q_k^0+j+s}$ since $q_k = q_k^0 + j$ for some $j \geq 0$.

Value of the objective function of the problem $(CP_{L_2}^{v+q_k^0})$ at the feasible solution Y is:

$$= \sum_{(i,j)\in \bigcup_{d=v+q_k^0+j+s}^{l_2} L_2^d} \lambda_d \, y_{ij}$$

$$< \sum_{(i,j)\in \bigcup_{d=v+q_k^0+j}^{l_2} L_2^d} \lambda_d \, x_{L_{2ij}}^{v+q_k^0}$$

(because of the nature of the order relations among $\lambda_j' s$)

But this cannot hold as $X_{L_2}^{v+q_k^0}$ is an optimal basic feasible solution of the problem $(CP_{L_2}^{v+q_k^0})$. Therefore, there does not exist $Y \in S$ such that $T_{L_2}(Y) = T_{L_2}^{v+q_k+s}$ and $T_{L_1}(Y) = T_{L_1}^{u-p_k}$.

To prove that $T_{L_1}^{u-p_k}$ is the minimum Level-I shipment time corresponding to the time $T_{L_2}^{v+q_k}$ of Level-II shipment, assume the contrary.

Therefore, there exists $Z \in S$ such that $T_{L_1}(Z) = T_{L_1}^{u-p_k+l} (< T_{L_1}^{u-p_k})$ and $T_{L_2}(Z) = T_{L_2}^{v+q_k}$.

Possibility-I. $T_{L_1}^{u-p_k+l} = T_{L_1}^{u-p_j}$ for some $j \in \{0, 1, 2, ..., k-1\}$ where, $p_0 = 0$.

But then $T_{L_2}(Z) = T_{L_2}^{v+q_k} < T_{L_2}^{v+q_j}$, which is not possible as $T_{L_2}^{v+q_j}$ is the minimum Level-II shipment time corresponding to time $T_{L_1}^{u-p_j}$ of Level-I shipment.

Possibility-II. $T_{L_1}^{u} < T_{L_1}^{u-p_k+l} < T_{L_1}^{u-p_k}$ and $T_{L_1}^{u-p_k+l} \neq T_{L_1}^{u-p_j}$ for any $j \in \{0, 1, 2, \ldots, k-1\}$.

This implies that there exists an interval $\left(T_{L_1}^{u-p_{m-1}}, T_{L_1}^{u-p_m}\right)$ such that $T_{L_1}^{u} < T_{L_1}^{u-p_{m-1}} < T_{L_1}^{u-p_k+l} < T_{L_1}^{u-p_m} < T_{L_1}^{u-p_k}$ where, $T_{L_1}^{u-p_{m-1}}$ and $T_{L_1}^{u-p_m}$ are two consecutive recorded Level-I shipment times in the first k recordings. Consider the time minimizing transportation problem $(TP_{L_2}^{v+q_{m-1}})$ whose optimal basic M-feasible solution is $X_{L_2}^{v+q_{m-1}}$. The Level-I and

Level-II shipment times at this M-feasible solution are: $T_{L_1}(X_{L_2}^{v+q_{m-1}}) = T_{L_1}^{u-p_m}$ and $T_{L_2}(X_{L_2}^{v+q_{m-1}}) = T_{L_2}^{v+q_m^0}$.

As $T_{L_2}^{v+q_k} < T_{L_2}^{v+q_{m-1}}$, Z is a feasible solution for the problem $(TP_{L_2}^{v+q_{m-1}})$ yielding Level-I and Level-II shipment times as $T_{L_1}^{u-p_k+l}$ and $T_{L_2}^{v+q_k}$ respectively.

Hence Z is a feasible solution of $(TP_{L_2}^{v+q_{m-1}})$ yielding its objective function value better than that yielded by $X_{L_2}^{v+q_{m-1}}$, which is not true as $X_{L_2}^{v+q_{m-1}}$ is an optimal basic M-feasible solution of $(TP_{L_2}^{v+q_{m-1}})$.

Hence $T_{L_1}^{u-p_k}$ is the minimum Level-I shipment time corresponding to the time $T_{L_2}^{v+q_k}$ of Level-II shipment. □

Remark 2.1. When optimal value of the objective function in the cost minimizing transportation problem $(CP_{L_2}^{v+q_k^0})$ is zero then for Level-I shipment times greater than $T_{L_1}^{u-p_k}$ the corresponding minimum Level-II shipment time would be zero. Therefore, the Level-I and Level-II shipment time pairs with Level-I shipment time greater than $T_{L_1}^{u-p_k}$ will not yield value of the objective function of the problem (HTP) smaller than $(T_{L_1}^{u-p_k} + 0)$. Hence Level-I shipment times greater than $T_{L_1}^{u-p_k}$ need not be investigated.

The next theorem (Theorem 2.3) characterizes the termination of the process of generating the pairs of the Level-I and the Level-II shipment times with Level-I shipment time greater than the Level-II shipment time.

Theorem 2.3. *Let an optimal basic feasible solution* $X_{L_2}^{v+q_{k-1}}$ *of the time minimizing transportation problem* $(TP_{L_2}^{v+q_{k-1}})$ *be not an M-feasible solution. Then, there does not exist any feasible solution* $X \in S$ *yielding the pair of the Level-I and Level-II shipment times as* $\big(T_{L_1}(X), T_{L_2}(X) : T_{L_1}(X) > T_{L_2}(X)\big)$ *such that*

$$\big(T_{L_1}(X) + T_{L_2}(X)\big) \leq \min_{j=0,1,\dots,k-1} \big(T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j}\big)$$

where, $p_0 = 0, q_0 \geq q_0^0 (= 0)$.

Proof. As $X_{L_2}^{v+q_{k-1}}$, an optimal basic feasible solution of the problem $(TP_{L_2}^{v+q_{k-1}})$, is not an M-feasible solution, $x_{L_{2ij}}^{v+q_{k-1}} > 0$ for some $(i, j) \in I \times J : t_{ij} = M$.

Therefore, by definition of $(TP_{L_2}^{v+q_{k-1}})$, $T_{L_2}(X_{L_2}^{v+q_{k-1}}) \geq T_{L_2}^{v+q_{k-1}}$.

Let $T_{L_2}(X_{L_2}^{v+q_{k-1}}) = T_{L_2}^{v+\hat{q}}$ $(\geq T_{L_2}^{v+q_{k-1}})$.

Case-I. $T_{L_1}(X_{L_2}^{v+q_{k-1}}) = T_{L_1}^{u-p_j}$ for some $j \in \{0, 1, 2, \ldots, k-1\}$.

$T_{L_1}(X_{L_2}^{v+q_{k-1}}) + T_{L_2}(X_{L_2}^{v+q_{k-1}}) = T_{L_1}^{u-p_j} + T_{L_2}^{v+\hat{q}}$.

As $T_{L_2}^{v+q_j}$ is the minimum time for Level-II shipment corresponding to time $T_{L_1}^{u-p_j}$ for Level-I shipment we have:

$$T_{L_1}^{u-p_j} + T_{L_2}^{v+\hat{q}} \geq T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j} \geq \min_{j=0,1,\ldots,k-1} \left[T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j}\right].$$

Case-II. $T_{L_1}(X_{L_2}^{v+q_{k-1}}) = T_{L_1}^{a}$ for some $a \in \{1, 2, \ldots, l_1\}$ such that $T_{L_1}^{u-p_0} < T_{L_1}^{a} < T_{L_1}^{u-p_{k-1}}$ and $T_{L_1}^{a} \neq T_{L_1}^{u-p_j}$ for any $j \in \{0, 1, \ldots, k-1\}$.

This implies that there exists a subinterval, say $\left[T_{L_1}^{u-p_{m-1}}, T_{L_1}^{u-p_m}\right]$ of $[T_{L_1}^{u-p_0}, T_{L_1}^{u-p_{k-1}}]$ such that $T_{L_1}^{u-p_{m-1}} < T_{L_1}^{a} < T_{L_1}^{u-p_m}$ Now,

$$\begin{aligned} T_{L_1}(X_{L_2}^{v+q_{k-1}}) + T_{L_2}(X_{L_2}^{v+q_{k-1}}) \quad &= T_{L_1}^{a} + T_{L_2}^{v+\hat{q}} \\ &> T_{L_1}^{u-p_{m-1}} + T_{L_2}^{v+\hat{q}}. \end{aligned}$$

As $T_{L_2}^{v+q_{m-1}}$ is the minimum time for Level-II shipment corresponding to time $T_{L_1}^{u-p_{m-1}}$ for Level-I shipment we have:

$$\begin{aligned} T_{L_1}^{u-p_{m-1}} + T_{L_2}^{v+\hat{q}} \quad &> T_{L_1}^{u-p_{m-1}} + T_{L_2}^{v+q_{m-1}} \\ &\geq \min_{j=0,..,k-1} (T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j}). \end{aligned}$$

Case-III. If $T_{L_1}(X_{L_2}^{v+q_{k-1}}) = T_{L_1}^{b} > T_{L_1}^{u-p_{k-1}}$ for some $b > 0$, then

$$\begin{aligned} T_{L_1}(X_{L_2}^{v+q_{k-1}}) + T_{L_2}(X_{L_2}^{v+q_{k-1}}) \quad &= T_{L_1}^{b} + T_{L_2}^{v+\hat{q}} \\ &\geq T_{L_1}^{b} + T_{L_2}^{v+q_{k-1}} \\ &> T_{L_1}^{u-p_{k-1}} + T_{L_2}^{v+q_{k-1}} \\ &\geq \min_{j=0,1,..,k-1} (T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j}). \end{aligned}$$

Hence the result. □

Remark 2.2. If the optimal basic feasible solution $X_{L_2}^{v+q_{k-1}}$ of the time minimizing transportation problem $(TP_{L_2}^{v+q_{k-1}})$ is not an M-feasible solution, there is no need of further generation of Level-I and Level-II shipment time pairs with Level-I shipment time greater than Level-II shipment time. The current upper bound on the optimal value of the objective function of the problem (HTP) would be $\min_{j=0,1,\ldots,k-1}\big(T_{L_1}^{u-p_j}+T_{L_2}^{v+q_j}:T_{L_1}^{u-p_j}>T_{L_2}^{v+q_j}\big)$.

To generate the pairs of the Level-I and Level-II shipment times with Level-I shipment time smaller than the Level-II shipment time, first the existence of a feasible solution with the Level-I shipment time smaller than $T_{L_2}^{v+q_0}$ and the Level-II shipment time greater than or equal to $T_{L_1}^{u}(\equiv T_{L_1}^{u-p_0})$ is examined. The next theorem (Theorem 2.4) pertains to the existence of such a solution.

Theorem 2.4. *Consider the following time minimizing transportation problem* $(TP_{L_1}^{v+q_0})$*:*

$$\min_{X\in S}\left\{\max_{I\times J}\big(\mathrm{t'}_{ij}(x_{ij})\big)\right\} \qquad (TP_{L_1}^{v+q_0})$$

where, $t'_{ij}=M(>>0),\ \forall\ (i,j)\in L_1:t_{ij}\geq T_{L_2}^{v+q_0}$.

If its optimal basic feasible solution, say $X_{L_1}^{v+q_0}$*, is M-feasible, then the Level-I shipment time* $T_{L_1}(X_{L_1}^{v+q_0})<T_{L_2}^{v+q_0}$ *and* $T_{L_2}(X_{L_1}^{v+q_0})\geq T_{L_1}^{u-p_0}$ $(p_0=0)$.

Proof. As the minimum overall shipment time in the time minimizing transportation problem (TP) is $T_{L_1}^{u}$, the proof follows directly from the definition of the problem $(TP_{L_1}^{v+q_0})$ and the nature of basic M-feasible solution. □

Remark 2.3. If optimal basic feasible solution of the problem $(TP_{L_1}^{v+q_0})$ is not an M-feasible solution, then there does not exist a feasible solution $X\in S$ of the problem (HTP) with the Level-I shipment time less than $T_{L_2}^{v+q_0}$.

Once the existence of the Level-I shipment time less than $T_{L_2}^{v+q_0}$ with the Level-II shipment time greater than or equal to $T_{L_1}^{u}$ is known, then it is pertinent to examine the existence of more pairs of Level-I and Level-II shipment times with Level-I shipment time smaller than the Level-II shipment time. Suppose that Level-I and Level-II shipment time pairs $\big(T_{L_1}^{u+\tilde{q}_{j-1}}, T_{L_2}^{v-\tilde{p}_{j-1}} : T_{L_1}^{u+\tilde{q}_{j-1}} < T_{L_2}^{v-\tilde{p}_{j-1}}\big)$, $j = 1, 2, \ldots, k$ are known. In each pair the Level-I shipment time is the minimum corresponding to the Level-II shipment time and also the Level-II shipment time is the minimum corresponding to the Level-I shipment time in that pair (Ref. Theorem 2.5). It may be noted that $\tilde{q}_j$'s and $\tilde{p}_j$'s for all $j = 1, 2, \ldots, k-1$ are non-negative integers such that $\tilde{p}_j > \tilde{p}_{j-1}$ and $\tilde{q}_j > \tilde{q}_{j-1}$. Therefore,

$$T_{L_1}^{u+\tilde{q}_0} > T_{L_1}^{u+\tilde{q}_1} > \ldots > T_{L_1}^{u+\tilde{q}_{k-1}} \text{ and } T_{L_2}^{v-\tilde{p}_0} < T_{L_2}^{v-\tilde{p}_1} < \ldots < T_{L_2}^{v-\tilde{p}_{k-1}}.$$

Consider the following time minimizing transportation problem $(TP_{L_1}^{u+\tilde{q}_{k-1}})$

$$\min_{X \in S} \Big\{ \max_{I \times J} \big(\mathfrak{t}'_{ij}(x_{ij})\big) \Big\} \qquad (TP_{L_1}^{u+\tilde{q}_{k-1}})$$

where, $t'_{ij} = M(>> 0),\ \forall\ (i,j) \in L_1 : t_{ij} \geq T_{L_1}^{u+\tilde{q}_{k-1}}$.

Suppose $X_{L_1}^{u+\tilde{q}_{k-1}}$ is its optimal basic M-feasible solution. By definition of M-feasible solution it follows that $T_{L_1}(X_{L_1}^{u+\tilde{q}_{k-1}}) < T_{L_1}^{u+\tilde{q}_{k-1}}$.

Let $T_{L_1}(X_{L_1}^{u+\tilde{q}_{k-1}}) = T_{L_1}^{u+\tilde{q}_k^0} (< T_{L_1}^{u+\tilde{q}_{k-1}})$. As in all the pairs $(T_{L_1}^{u+\tilde{q}_j}, T_{L_2}^{v-\tilde{p}_j})$, $j = 0, 1, \ldots, k-1$, $T_{L_1}^{u+\tilde{q}_j}$ is the minimum Level-I shipment time corresponding to the time $T_{L_2}^{v-\tilde{p}_j}$ of the Level-II shipment, it follows that $T_{L_2}(X_{L_1}^{u+\tilde{q}_{k-1}}) > T_{L_2}^{v-\tilde{p}_{k-1}}$. Let $T_{L_2}(X_{L_1}^{u+\tilde{q}_{k-1}}) = T_{L_2}^{v-\tilde{p}_k} (> T_{L_2}^{v-\tilde{p}_{k-1}})$. The next theorem (Theorem 5) pertains to the generation of the pair $(T_{L_1}^{u+\tilde{q}_k}, T_{L_2}^{v-\tilde{p}_k})$ of the Level-I and Level-II shipment times. In this pair $T_{L_1}^{u+\tilde{q}_k}$ is the minimum Level-I shipment time corresponding to the time $T_{L_2}^{v-\tilde{p}_k}$ of the Level-II shipment and $T_{L_2}^{v-\tilde{p}_k}$ is the minimum Level-II shipment time corresponding to the time $T_{L_1}^{u+\tilde{q}_k}$ of the Level-I shipment. This theorem involves the following cost minimizing transportation problem (named as $(CP_{L_1}^{u+\tilde{q}_k^0})$).

$$\min_{X \in S} \sum_{I \times J} c_{ij} x_{ij} \qquad (CP_{L_1}^{u+\tilde{q}_k^0})$$

where,

$$
\begin{aligned}
c_{ij} &= M, && (i,j) \in L_1 : t_{ij} > T_{L_1}^{u+\tilde{q}_k^0}, \text{ or } (i,j) \in L_2 : t_{ij} > T_{L_2}^{v-\tilde{p}_k} \\
&= 0, && (i,j) \in L_2 : t_{ij} \leq T_{L_2}^{v-\tilde{p}_k} \\
&= \lambda_{u+\tilde{q}_k^0+r}, && (i,j) \in L_1 : t_{ij} = T_{L_1}^{u+\tilde{q}_k^0+r}, r = 0,1,\ldots,(l_1-(u+\tilde{q}_k^0))
\end{aligned}
$$

λ_j's being positive integers such that $\lambda_j >> \lambda_{j+1}\ \forall\ j$.

Theorem 2.5. *If $X_{L_1}^{u+\tilde{q}_k^0}$ is an optimal basic feasible solution of the cost minimizing transportation problem $(CP_{L_1}^{u+\tilde{q}_k^0})$, then $T_{L_1}(X_{L_1}^{u+\tilde{q}_k^0})$ is the minimum Level-I shipment time corresponding to the time $T_{L_2}^{v-\tilde{p}_k}$ of the Level-II shipment. Also $T_{L_2}^{v-\tilde{p}_k}$ is the minimum Level-II shipment time corresponding to the time $T_{L_1}(X_{L_1}^{u+\tilde{q}_k^0})$ of the Level-I shipment.*

Proof. The proof runs exactly on the lines of the Theorem 2. In place of the problems $(CP_{L_2}^{v+q_k^0})$ and $(TP_{L_2}^{v+q_{k-1}})$ proof will now depend upon the use of the problems $(CP_{L_1}^{u+\tilde{q}_k^0})$ and $(TP_{L_1}^{u+\tilde{q}_{k-1}})$ respectively. □

The following theorem (Theorem 2.6) characterizes the termination of the process of searching of more of Level-I and Level-II shipment time pairs with Level-I shipment time less than the Level-II shipment time.

Theorem 2.6. *If optimal basic feasible solution $X_{L_1}^{u+\tilde{q}_k}$ of the time minimizing transportation problem $(TP_{L_1}^{u+\tilde{q}_k})$ is not an M-feasible solution, then there does not exist a Level-I and Level-II shipment time pair $\Big((T_{L_1}(.), T_{L_2}(.)) : T_{L_2}(.) > T_{L_1}(.)\Big)$ such that*

$$T_{L_1}(.) + T_{L_2}(.) < \min_{j=\{0,1,\ldots,k\}} \big(T_{L_1}^{u+\tilde{q}_j} + T_{L_2}^{v-\tilde{p}_j}\big).$$

Proof. Proof is similar to the proof of Theorem 2.3 except that in place of the problem $(TP_{L_2}^{v+q_k})$ one will have to use the problem $(TP_{L_1}^{u+\tilde{q}_k})$. □

Remark 2.4. Whenever the time minimizing transportation problem $(TP_{L_1}^{u+\tilde{q}_k})$ does not have a basic M-feasible solution then, the best Level-I and Level-II shipment time pair with Level-I shipment time smaller than the Level-II shipment time correspond to

$$\min_{j=0,1,\ldots,k} \left(T_{L_1}^{u+\tilde{q}_j} + T_{L_2}^{v-\tilde{p}_j}\right)$$

and therefore, the Level-I and Level-II shipment time pairs beyond the pair $\left(T_{L_1}^{u+\tilde{q}_k} + T_{L_2}^{v-\tilde{p}_k}\right)$ need not be studied.

The last theorem (Theorem 2.7) given below characterizes the global minimizer of the two level hierarchical time minimizing transportation problem (HTP).

Theorem 2.7. *The global minimum value of the objective function of the two level hierarchical time minimizing transportation problem* (HTP) *is*

$$\min\left\{\begin{array}{l} \min\limits_{j\geq 0}\left(T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j} : T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j}\right), \\ \min\limits_{j\geq 0}\left(T_{L_1}^{u+\tilde{q}_j} + T_{L_2}^{v-\tilde{p}_j} : T_{L_1}^{u+\tilde{q}_j} < T_{L_2}^{v-\tilde{p}_j}\right) \end{array}\right\}$$

where, p_j*'s,* q_j*'s,* $\tilde{q}_j$*'s and* $\tilde{p}_j$*'s are non-negative integers such that* $p_j > p_{j-1}$, $q_j > q_{j-1}$, $\tilde{q}_j > \tilde{q}_{j-1}$ *and* $\tilde{p}_j > \tilde{p}_{j-1}$ *for all* $j \geq 1,\ p_0 = 0,\ q_0 \geq 0$.

Proof. Let

$$\min\left\{\begin{array}{l} \min\limits_{j\geq 0}\left(T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j} : T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j}\right), \\ \min\limits_{j\geq 0}\left(T_{L_1}^{u+\tilde{q}_j} + T_{L_2}^{v-\tilde{p}_j} : T_{L_1}^{u+\tilde{q}_j} < T_{L_2}^{v-\tilde{p}_j}\right) \end{array}\right\}$$

$$= T_{L_1}^{u-p_s} + T_{L_2}^{v+q_s} \text{ (say) for some } s \geq 0.$$

If possible, assume that the statement of the theorem is not true.

This implies that there exists $\hat{X} \in S$ such that

$$T_{L_1}(\hat{X}) + T_{L_2}(\hat{X}) < T_{L_1}^{u-p_s} + T_{L_2}^{v+q_s}. \tag{2.1}$$

As $\min\limits_{S}\{\max\limits_{I\times J} \mathfrak{t}_{ij}(x_{ij})\} = T_{L_1}^{u}$, $T_{L_1}(\hat{X}) \geq T_{L_1}^{u}$.

Let $|\{(T_{L_1}^{u-p_j}, T_{L_2}^{v+q_j}) : T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j}, j \geq 0\}| = h+1$. That is, $(T_{L_1}^{u-p_h}, T_{L_2}^{v+q_h})$ is the last pair in the set $\{(T_{L_1}^{u-p_j}, T_{L_2}^{v+q_j})$: $T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j}, j \geq 0\}$.

Case-I. $T_{L_1}^{u-p_0} \leq T_{L_1}(\hat{X}) \leq T_{L_1}^{u-p_h}$

(a) $T_{L_1}(\hat{X}) = T_{L_1}^{u-p_k}$ for some $k \in \{0, 1, 2, 3, ..., h\}$.
Since $T_{L_1}(\hat{X}) + T_{L_2}(\hat{X}) < T_{L_1}^{u-p_s} + T_{L_2}^{v+q_s} \leq T_{L_1}^{u-p_k} + T_{L_2}^{v+q_k}$, it follows that $T_{L_2}(\hat{X}) < T_{L_2}^{v+q_k}$.
Let $T_{L_2}(\hat{X}) = T_{L_2}^{v+q_k+j} (< T_{L_2}^{v+q_k})$ for some $j \geq 1$.
This implies that $\hat{X}$ is a feasible solution of the cost minimizing transportation problem $(CP_{L_2}^{v+q_k^0})$ because $T_{L_2}^{v+q_k} < T_{L_2}^{v+q_k^0}$.
Now, the value of the objective function of the problem $(CP_{L_2}^{v+q_k^0})$ at this feasible solution $\hat{X}$ is:

$$= \sum_{(i,j) \in \bigcup\limits_{d=v+q_k+j}^{l_2} L_2^d} \lambda_d \; \hat{x}_{ij}$$

$$< \sum_{(i,j) \in \bigcup\limits_{d=v+q_k^0}^{l_2} L_2^d} \lambda_d \; x_{L_{2ij}}^{u+q_k^0} \quad \text{since } q_k + j > q_k \geq q_k^0$$

which can not hold as $X_{L_1}^{u+q_k^0}$ is an optimal feasible solution of the problem $(CP_{L_2}^{v+q_k^0})$.

(b) $T_{L_1}(\hat{X}) = T_{L_1}^{u-\hat{p}}$ (say) and $T_{L_1}^{u-\hat{p}} \neq T_{L_1}^{u-p_j}$ for any j.
This implies that there exists an index, say f, such that $T_{L_1}^{u-p_{f-1}} < T_{L_1}^{u-\hat{p}} < T_{L_1}^{u-p_f}$.
Since $T_{L_1}^{u-\hat{p}} = T_{L_1}(\hat{X}) > T_{L_1}^{u-p_{f-1}}$, we have $T_{L_2}(\hat{X}) < T_{L_2}^{v+q_{f-1}}$. Therefore, $\hat{X}$ is an M-feasible solution of the problem $(TP_{L_2}^{v+q_{f-1}})$ yielding its objective value $T_{L_1}^{u-\hat{p}} < T_{L_1}^{u-p_f}$. This contradicts the fact that $T_{L_1}^{u-p_f}$ is the optimal value of the objective function in the problem $(TP_{L_2}^{v+q_{f-1}})$.

Case-II. $T_{L_1}(\hat{X}) > T_{L_1}^{u-p_h}$.

As $(T_{L_1}^{u-p_h}, T_{L_2}^{v+q_h})$ is the last pair in the set $\{(T_{L_1}^{u-p_j}, T_{L_2}^{v+q_j}) : T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j},\ j \geq 0\}$, it follows from Remark 2.1 on Theorem 2.2 and Remark 2.2 on Theorem 2.3 that either optimal value of the objective function of the cost minimizing transportation problem $(CP_{L_2}^{v+q_h^0})$ is zero or optimal basic feasible solution of the time minimizing transportation problem $(TP_{L_2}^{v+q_h})$ is not an M-feasible solution.

(a) The optimal value of the objective function of the cost minimizing transportation problem $(CP_{L_2}^{v+q_h^0})$ is zero.
In this case, by virtue of Remark 2.1 on Theorem 2.2 it follows that $T_{L_2}^{v+q_h} \equiv T_{L_2}(X_{L_2}^{v+q_h^0}) = 0$.
Therefore, $T_{L_1}(\hat{X}) + T_{L_2}(\hat{X}) > T_{L_1}^{u-p_h} + T_{L_2}^{v+q_h}(= 0)$, which contradicts (2.1).

(b) An optimal basic feasible solution of the time minimizing transportation problem $(TP_{L_2}^{v+q_h})$ is not an M-feasible solution.
Since we have $T_{L_1}(\hat{X}) > T_{L_1}^{u-p_h}$, it follows that $T_{L_2}(\hat{X}) < T_{L_2}^{v+q_h}$.
But this implies that $\hat{X}$ is an M-feasible solution of the problem $(TP_{L_2}^{v+q_h})$, which is not possible because this problem in the current case has no M-feasible solution.

Hence there does not exist $\hat{X}$ satisfying

$$T_{L_1}(\hat{X}) + T_{L_2}(\hat{X}) < \min_{j\geq 0}\left(T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j} : T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j}\right).$$

Similarly if

$$\min\left\{\begin{array}{l}\min\limits_{j\geq 0}\left(T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j}:\ T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j}\right),\\ \min\limits_{j\geq 0}\left(T_{L_1}^{u+\tilde{q}_j} + T_{L_2}^{v-\tilde{p}_j}:\ T_{L_1}^{u+\tilde{q}_j} < T_{L_2}^{v-\tilde{p}_j}\right)\end{array}\right\} = \min_{j\geq 0}\left(T_{L_1}^{u+\tilde{q}_j} + T_{L_2}^{v-\tilde{p}_j}\right)$$

one can establish the result likewise. □

Remark 2.5. If

$$\min\left\{\begin{array}{l}\min\limits_{j\geq 0}\left(T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j} : T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j}\right),\\ \min\limits_{j\geq 0}\left(T_{L_1}^{u+\tilde{q}_j} + T_{L_2}^{v-\tilde{p}_j} : T_{L_1}^{u+\tilde{q}_j} < T_{L_2}^{v-\tilde{p}_j}\right)\end{array}\right\} = T_{L_1}^{u-p_k} + T_{L_2}^{v+q_k},$$

then the optimal basic feasible solution $X_{L_2}^{v+q_k^0}$ of the cost minimizing transportation problem $(CP_{L_2}^{v+q_k^0})$ is a global minimizer of the two level hierarchical time minimizing transportation problem (HTP).

3 Algorithm

Step 1 Obtain optimal basic feasible solution of the problem (TP). If its optimal basic feasible solution consists of only Level-I links or Level-II links, then stop and go to step 7.

Else, note the corresponding Level-I shipment time as $T^{u}_{L_1}$ and Level-II shipment time as $T^{v}_{L_2}$. ($T^{u}_{L_1} \geq T^{v}_{L_2}$, say).

Construct the cost minimizing transportation problem $(CP^{v+q^0_0}_{L_2}), q^0_0 = 0$ and solve it to find the minimum Level-II shipment time corresponding to the time $T^{u}_{L_1} (\equiv T^{u-p_0}_{L_1})$ of Level-I shipment. Record this pair as $(T^{u-p_0}_{L_1}, T^{v+q_0}_{L_2}), p_0 = 0, q_0 \geq 0$.

If $T^{u-p_0}_{L_1} = T^{1}_{L_1}$ or the optimal value of the objective function of the problem $(CP^{v+q^0_0}_{L_2})$ is zero, go to step 3.

Else, go to step 2.

Step 2 ($\mathbf{k \geq 1}$) Construct the time minimizing transportation problem $(TP^{v+q_{k-1}}_{L_2})$. Find its optimal basic feasible solution. If it is not an M-feasible solution, then go to step 3. Else, construct the cost minimizing transportation problem $(CP^{v+q^0_k}_{L_2})$ and find its optimal basic feasible solution. If its optimal value is non-zero, then read the corresponding Level-I and Level-II shipment times. Record this pair as $(T^{u-p_k}_{L_1}, T^{v+q_k}_{L_2})$.

If $T^{u-p_k}_{L_1} = T^{1}_{L_1}$ or optimal value of the objective function of the cost minimizing transportation problem $(CP^{v+q^0_k}_{L_2})$ is zero, then no further useful Level-I and Level-II shipment time pair with Level-I shipment time greater than the Level-II shipment time can be constructed and hence go to step 3.

Else, go to step 5.

Step 3 Construct the time minimizing transportation problem $(TP^{v+q_0}_{L_1})$, find its optimal basic feasible solution, if it is not an M-feasible solution, then go to step 7.

Else, read the corresponding Level-I and Level-II shipment times as $T^{u+\tilde{q}^0_0}_{L_1}$ and $T^{v-\tilde{p}_0}_{L_2}$. Construct the cost minimizing transportation problem $(CP^{u+\tilde{q}^0_0}_{L_1})$, find its optimal basic feasible solution. If optimal value of its objective function is non-zero, read the corresponding shipment times of both the levels. Record this pair as $(T_{L_1}^{u+\tilde{q}_0}, T_{L_2}^{v-\tilde{p}_0})$.

If $T_{L_2}^{v-\tilde{p}_0} = T_{L_2}^1$ or the optimal value of the objective function of the problem $(CP_{L_1}^{u+\tilde{q}_0^0})$ is zero, then no further useful Level-I and Level-II shipment time pair with Level-I shipment time less than the Level-II shipment time can be constructed and hence go to step 7. Else, go to step 4.

Step 4 $(\mathbf{j \geq 1})$ Construct the problem $(TP_{L_1}^{u+\tilde{q}_{j-1}})$ and find its optimal basic feasible solution. If it is not an M-feasible solution, then no more useful Level-I and Level-II shipment time pair with Level-I shipment time smaller than the Level-II shipment time can be constructed and hence go to step 7. Else, note the Level-I and Level-II shipment times as $T_{L_1}^{u+\tilde{q}_j^0}$ and $T_{L_2}^{v-\tilde{p}_j}$ respectively. Construct the cost minimizing transportation problem $(CP_{L_1}^{u+\tilde{q}_j^0})$, find its optimal basic feasible solution. If its optimal value is non-zero, then read the corresponding Level-I and Level-II shipment times. Record this Level-I and Level-II shipment time pair as $(T_{L_1}^{u+\tilde{q}_j}, T_{L_2}^{v-\tilde{p}_j})$.

If $T_{L_2}^{v-\tilde{p}_j} = T_{L_2}^1$ or optimal value of the objective function of the problem $(CP_{L_1}^{u+\tilde{q}_j^0})$ is zero, go to step 7.

Else, go to step 6.

Step 5 Execute step 2 for next higher value of k.

Step 6 Execute step 4 for next higher value of j.

Step 7 Find

$$\min \left\{ \begin{array}{l} \min\limits_{j \geq 0} \left(T_{L_1}^{u-p_j} + T_{L_2}^{v+q_j} : \ T_{L_1}^{u-p_j} > T_{L_2}^{v+q_j}\right), \\ \min\limits_{j \geq 0} \left(T_{L_1}^{u+\tilde{q}_j} + T_{L_2}^{v-\tilde{p}_j} : \ T_{L_1}^{u+\tilde{q}_j} < T_{L_2}^{v-\tilde{p}_j}\right) \end{array} \right\}$$

This will be the optimal value of the objective function of the two level hierarchical time minimizing transportation problem.

4 Numerical Illustration

Consider the following 6×8 two level hierarchical time minimizing transportation problem, where shaded cells denote the Level-I links.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	5	3	7	9	5	1	10	6	9
S_2	13	4	6	12	12	10	9	3	2
S_3	8	13	2	9	3	8	9	6	2
S_4	4	1	4	4	9	6	13	13	10
S_5	2	6	2	6	13	12	5	12	6
S_6	9	10	4	8	7	6	4	7	6
b_j	5	8	6	2	6	3	2	3	

$T_{L_1}^1 = 13,\ T_{L_1}^2 = 12,\ T_{L_1}^3 = 10,\ T_{L_1}^4 = 9,\ T_{L_1}^5 = 8,\ T_{L_1}^6 = 7,\ T_{L_1}^7 = 6$ $T_{L_1}^8 = 4,\ T_{L_1}^9 = 3,\ T_{L_1}^{10} = 2,\ T_{L_1}^{11} = 1$; therefore, $l_1 = 11$.

$T_{L_2}^1 = 13,\ T_{L_2}^2 = 12,\ T_{L_2}^3 = 10,\ T_{L_2}^4 = 9,\ T_{L_2}^5 = 8,\ T_{L_2}^6 = 7,\ T_{L_2}^7 = 6,$ $T_{L_2}^8 = 5,\ T_{L_2}^9 = 4,\ T_{L_2}^{10} = 3,\ T_{L_2}^{11} = 2$; therefore, $l_2 = 11$.

An optimal basic feasible solution of (TP) yields the value of Level-I shipment time as $T_{L_1}^u (= T_{L_1}^{u-p_0}) = 6$ and Level-II shipment time as $T_{L_2}^v (= T_{L_2}^{v+q_0^0}) = 5$.

An optimal basic feasible solution of the cost minimizing transportation problem $(CP_{L_2}^{v+q_0^0})$ yields Level-I shipment time as 6 units and the corresponding minimum Level-II shipment time as 5 units. Hence $T_{L_2}^{v+q_0} = 5$.

The first recorded pair is (6,5).

Next, time minimizing transportation problem $(TP_{L_2}^{v+q_0})$ is constructed whose optimal basic feasible solution is an M-feasible solution yielding Level-I and Level-II shipment times respectively as $T_{L_1}^{u-p_1} = 7$ and $T_{L_2}^{v+q_1^0} = 4$.

The optimal basic feasible solution of cost minimizing transportation problem $(CP_{L_2}^{v+q_1^0})$ yields the minimum Level-II shipment time as $T_{L_2}^{v+q_1} = 4$ corresponding to the Level-I shipment time $T_{L_1}^{u-p_1} = 7$.

The second recorded pair is (7,4).

Next, time minimizing transportation problem $(TP_{L_2}^{v+q_1})$ is constructed whose optimal basic feasible solution is an M-feasible solution yielding Level-I and Level-II shipment times respectively as $T_{L_1}^{u-p_2} = 9$ and $T_{L_2}^{v+q_2^0} = 3$.

An optimal basic feasible solution of the cost minimizing transportation problem $(CP_{L_2}^{v+q_2^0})$ yields the minimum Level-II shipment time as $T_{L_2}^{v+q_2} = 2$ corresponding to the Level-I shipment time $T_{L_1}^{u-p_2} = 9$.

The third recorded pair is (9,2).

Next, construct the time minimizing transportation problem $(TP_{L_2}^{v+q_2})$. As its optimal basic feasible solution is an M-feasible solution, record Level-I shipment time as $T_{L_1}^{u-p_3} = 10$ and for this solution $x_{ij} = 0 \; \forall \, (i,j) \in L_2$. This implies that no shipment is done over source-destination links in Level-II and hence Level-II shipment time is taken as zero.

The fourth recorded pair is (10,0).

As the Level-II shipment time is zero in the pair (10,0), go to step 3. The time minimizing transportation problem $(TP_{L_1}^{v+q_0})$ is formed and its optimal basic feasible solution is obtained. We get $T_{L_1}^{u+\tilde{q}_0^0} = 4$ and $T_{L_2}^{v-\tilde{p}_0} = 7$.

The cost minimizing transportation problem $(CP_{L_1}^{u+\tilde{q}_0^0})$ is constructed, whose optimal basic feasible solution yields the minimum Level-I shipment time as $T_{L_1}^{u+\tilde{q}_0} = 2$ corresponding to the Level-II shipment time $T_{L_2}^{v-\tilde{p}_0} = 7$.

Hence the fifth recorded pair is (2,7).

Next, the time minimizing transportation problem $(TP_{L_1}^{u+\tilde{q}_0})$ is constructed, whose optimal basic feasible solution yields Level-I and Level-II shipment times respectively as $T_{L_1}^{u+\tilde{q}_1^0} = 1$ and $T_{L_2}^{v-\tilde{p}_1} = 8$.

The optimal basic feasible solution of the cost minimizing transportation problem $(CP_{L_1}^{u+\tilde{q}_1^0})$ yields the minimum Level-I shipment time as $T_{L_1}^{u+\tilde{q}_1} = 1$ corresponding to the Level-II shipment time $T_{L_2}^{v-\tilde{p}_1} = 8$.

The sixth recorded pair is (1,8).

Next, the time minimizing transportation problem $(TP_{L_1}^{u+\tilde{q}_1})$ is constructed whose optimal feasible solution is not an M-feasible solution, hence we stop and go to the terminal step.

The optimal value of the objective function of the problem (HTP) is:

$$\min\Big((6+5),\ (7+4),\ (9+2),\ (10+0);\ (2+7),\ (1+8)\Big) = 9.$$

Optimal shipment schedules for Level-I and Level-II decision makers are given below. (It may be noted that in this example alternate optimal feasible solution exists.)

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	a_i
S_1	② 5	3	① 7	9	⑥ 5	1	10	6	9
S_2	13	② 4	6	12	12	10	9	3	2
S_3	8	13	② 2	9	3	8	9	6	2
S_4	4	⑤ 1	4	② 4	9	③ 6	13	13	10
S_5	③ 2	① 6	2	6	13	12	② 5	12	6
S_6	9	10	③ 4	8	7	6	4	③ 7	6
b_j	5	8	6	2	6	3	2	3	

5 Concluding Remarks

(a) In hierarchical optimization problems there is a hierarchical ordering of decision makers, and one set has the authority to strongly influence the preferences of the other decision makers. Such situations are analyzed using Stackelberg strategy (Baser and Olsder (1982), Baser and Selbuz (1979)).

Hierarchical optimization has found applications in areas like defense (Bracken and McGill (1974)), competitive economies (Bracken and McGill (1978)), government regulations (Bard (1983) Bard (1984)), equipment scheduling (Aoki and Satoh (1982)), decentralized control (Bard, 1983), and imperfectly competitive spatial economies and equilibrium facility locations (Friesz et al. (1988), Tobin and Friesz (1986)).

Transportation network design has especially been an active application area of hierarchical optimization techniques. For example, hierarchical optimization in the transport field finds its applications in system planning (Fisk (1986)), signal optimization (Marcotte (1983)), and network design (Friesz (1985), Suwansirikul et al. (1987)).

In the Two-Level Hierarchical Time Minimizing Transportation Problem studied in this paper, there are some sensitive source-destination links forming the set L_1. These sensitive links are used for partial shipment by the planner (called Level-I decision-maker). Corresponding to a planner's shipment schedule, the transporter (called Level-II decision-maker), using only Level-II links, makes the best decision for his shipment to meet the left over demand of the destinations. The planner aims at finding that feasible shipment schedule over the sensitive links in L_1 for which the corresponding optimal feasible schedule of the transporter in Level-II is such that the sum of shipment times in the two levels is the least. This is a very special case of Mathematical Programming Problems with Equilibrium Constraints which, in general, are hard problems. Even though the Two Level Hierarchical Time Minimizing Transportation Problem is a concave minimization problem (CMP), a successful attempt has been made to develop a polynomial bound algorithm based upon sound mathematical results established in this paper. This problem has been studied by associating it to a well-studied standard time minimizing transportation problem. Standard time minimizing transportation problem is also a (CMP) for which the algorithms proposed by Hammer (Hammer (1969)) and Garfinkel and

Rao (Garfinkel and Rao (1971)) are all polynomial bound algorithms. To accelerate the convergence, the proposed algorithm uses domain reduction in form of source-destination links abandonment.

(b) To generate the Level-I and Level-II shipment time pairs with Level-I shipment time greater than Level-II shipment time, at most $(l_2 - v)$ balanced time minimizing transportation problems and associated cost minimizing transportation problems are to be solved. Similarly to obtain the Level-I and Level-II shipment time pairs with Level-I shipment time less than Level-II shipment time, not more than $(l_1 - u)$ balanced time minimizing transportation problems and the associated cost minimizing transportation problems are to be solved. As time minimizing transportation problem and cost minimizing transportation problem are solvable by polynomial bound algorithm and as only finite number of such problems is to be investigated in the proposed approach, it follows that the developed algorithm is a polynomial bound algorithm. Abandoning of various source-destination links at the successive time minimizing transportation problems and cost minimizing transportation problems accelerate the convergence of the algorithm. Otherwise also one may use network based-approach and simply do not include the forbidden arcs into the bipartite graph representing a cost minimizing transportation problem. It is well known that the sparse network flow problems can be solved much faster than the dense ones.

(c) The proposed algorithm has been coded in C++ and verified successfully with the help of lot of examples of various sizes. Recording of some such examples is listed Table 1. (The code of the proposed algorithm can be made available, as and when desired.)

About 30 problems of each of above mentioned sizes have been solved, each yielding the optimal solution within seconds on a Pentium-4 powered linux workstation.

(d) The study presented in this paper can be extended to accommodate multi-index transportation problem, bulk transportation problem and time minimizing assignment problem.

Size of	No. of		No. of		No. of		No. of	Optimal	Optimal
the	links in		partitions in		TMTPs	CMTPs	pairs	pair(s)	value
problem	Level	Level	Level	Level	solved	solved	obtained		
	I	II	I	II	solved	solved	obtained		
4×6	12	12	7	7	4	3	3	(3,11)	14
5×5	13	12	11	6	4	3	3	(8,4)	12
6×8	27	21	12	12	7	6	6	(2,7), (1,8)	9
7×6	23	19	10	11	4	3	3	(6,5),(1,10)	11
7×9	28	35	11	13	5	4	4	(2,9),(1,10)	11
8×8	29	35	13	12	4	3	3	(10,2)	12
8×9	38	34	13	13	6	6	6	(0,10)	10
8×10	42	38	11	13	5	4	4	(1,7)	8
9×8	36	36	12	13	4	4	4	(12,2)	14
9×9	35	46	10	13	4	4	4	(0,13)	13
10×8	42	38	12	13	5	5	5	(6,4)	10
10×10	59	41	13	13	5	4	4	(7,5),(12,0)	12
11×7	34	43	13	12	5	3	3	(4,7)	11
11×8	45	43	13	11	5	4	4	(0,12)	12
12×5	28	32	11	13	5	3	3	(6,9),(12,3)	15
12×9	49	59	13	13	5	4	4	(0,13)	13
13×7	37	54	13	12	7	6	6	(6,5)	11
13×9	59	58	13	13	6	4	3	(2,10)	12
14×6	47	37	13	13	6	6	6	(3,8)	11
15×7	51	54	13	12	5	5	6	(6,5),(8,3)	11
15×10	79	71	13	13	8	7	7	(2,8)	10
20×5	51	49	13	12	5	4	4	(12,2),(13,1)	14

Table 1

References

Ahuja R.K., Orlin J.B., Stein C. and Tarjan R.E. (1994). Improved Algorithms for Bipartite Network Flow. *SIAM Journal of Computing* 23, 906-933.

Aoki K. and Satoh T. (1982). Economic Dispatch with Network Security Constraints using Parametric Quadratic Programming, *IEEE Transactions on Power Apparatus and Systems* PAS-101, 4548-4556.

Bansal S. and Puri M.C. (1980). A Min-Max Problem, *ZOR* 24, 191-200.

Bard J.F. (1983). Coordination of Multidivisional Firm Through Two Levels of Management. *OMEGA* 11, 457-465.

Bard J.F. (1983/84). Regulating Nonnuclear Industrial Waste by Hazard Classification. *Journal of Environmental Systems* 13, 21-41.

Basar T. and Olsder G.J. (1982). *Dynamic Noncooperative Games*. Academic Press.

Basar T. and Selbuz H. (1979). Closed Loop Stackelberg Strategies with Applications to Optimal Control of Multi-level Systems. *IEEE Transactions Automatic Control* 24, 166-178.

Bracken J. and McGill J.M. (1974). Defense Applications of Mathematical Programs with Optimization Problems in Constraints. *Operations Research* 22, 1086-1096.

Bracken J. and McGill J.M. (1978). Production and Marketing Decisions with Multiple Objectives in a Competitive Environment. *Journal of Optimization Theory and Applications* 24, 449-458.

Fisk C.S. (1986). A Conceptual Framework for Optimal Transportation Systems Planning with Integrated Supply and Demand Models. *Transportation Science* 20, 37-47.

Friesz T.L. (1985). Transportation Network Equilibrium, Design and Aggregation. *Transportation Research.* 19A, 413-427.

Friesz T.L., Tobin R.L. and Miller T. (1988). Theory and Algorithms for Equilibrium Network Facility Location. *Environment and Planning B* 15, 191-203.

Garfinkel R.S. and Rao M.R. (1971). Bottleneck Transportation Problem. *Naval Research Logistics Quarterly* 18, 465-472.

Hammer P.L. (1969). Time Minimization Transportation Problem. *Naval Research Logistics Quarterly* 16, 345-357.

Horst R. and Thoai N.V. (1998). An Integer Concave Minimization Approach for the Minimum Concave Cost Capacitated Flow Problem on Networks. *OR*

Spektrum 20, 47-53.

Kleinschmidt P.K. and Schannath H. (1995). A Strongly Polynomial Algorithm for the Transportation Problem. *Mathematical Programming* 68, 1-13.

Klincewicz John G. (1990). Solving a Freight Transportation Problem using Facility Location Techniques. *Operations Research* 38, 99-109.

Kuno T. and Utsunomiya T. (1996). A Decomposition Algorithm for Solving Certain Classes of Production-Transportation Problems with Concave Production Cost. *Journal of Global Optimization* 8, 67-80.

Kuno T. and Utsunomiya T. (2000). A Lagrangian Based Branch-and-Bound Algorithm for Production Transportation Problem. *Journal of Global Optimization* 18, 59-73.

Locatelli M. and Nguyan V.T. (2000). Finite Exact Branch and Bound Algorithm for Concave Minimization over Polytopes. *Journal of Global Optimization* 18, 107-128.

Marcotte P. (1983). Network Optimization with Continuous Control Parameters. *Transportation Science* 17, 181-197.

Orlin J.B. (1988). A Faster Strongly Polynomial Minimum Cost Flow Algorithm. Proceeding of the 20th ACM Symposium on the Theory of Computing 11, 377-387.

Sharma R.R.K. and Sharma K.D. (2000). A New Dual-Based Procedure for the Transportation Problem. *European Journal of Operational Research* 122, 611-624.

Sherali H.D. (1982). Equivalent Weights for Lexicographic Multi-Objective Programs Characterization and Computations. *European Journal of Operational Research* 11, 367-379.

Suwansirikul C., Friesz T.L. and Tobin R.L. (1987). Equilibrium Decomposed Optimization: A heuristic for the Continuous Equilibrium Network Design Problem. *Transportation Science* 21, 254-263.

Szwarc W. (1971). Some Remarks on the Time Transportation Problem. *Naval Research Logistics Quarterly* 18, 473-485.

Tardos E. (1985). A Strongly Polynomial Minimum Cost Circulation Algorithm. *Combinatorica* 5, 247-255.

Tardos E. (1986). A Strongly Polynomial Algorithm to Solve Combinatorial Linear Programming. *Operations Research* 34, 250-256.

Tobin T.L. and Friesz T.L. (1986). Spatial Competition Facility Location Models. *Annals of Operations Research* 6, 49-74.

Tuy H., Ghannadan S., Migdalays A. and Vărbrand P. (1996). A Strongly Polynomial Algorithm for a Concave Production-Transportation Problem with a Fixed Number of Nonlinear Variables. *Mathematical Programming* 72, 229-258.

Top

Volume 12, Number 2
December 2004

CONTENTS

Page