ON A STANDARD TIME TRANSPORTATION PROBLEM

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ABSTRACT

This paper studies a standard time transportation problem. Parallel transportation is done from all the sources but a source supplying goods to more than one destination can ship to another destination only after the earlier chosen destinations have been served. The aim is to minimize the maximum of the total time that the various sources take to serve the various destinations. A lexi-search algorithm is proposed to obtain a global optimal solution. A heuristic is also explained to obtain a near optimal starting upper bound on the value of the objective function. The lexi search is further facilitated by establishing some results in form of theorems and remarks.

KEYWORDS : combinatorial optimization, transportation problem, lexi-search.

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1. INTRODUCTION

This paper studies a standard time transportation problem. Let $I = \{1,2,\ldots,m\}$ be the index set of m sources and $J = \{1,2,\ldots,n\}$ be the index set of n destinations. Let a_i be the available supply at the source $i \in I$ and b_j, $j \in J$ be the demand of the j^{th} destination. t_{ij} be the time of transporting the goods from the i^{th} source to the j^{th} destination, $1 \leq i \leq m$, $1 \leq j \leq n$ and, it is independent of the quantity transported. Each destination can receive its requirement from any number of sources and a source can supply goods to any number of destinations subject to its capacity restrictions. None of the sources is to be left unutilized and none of the destinations is to be left unserved and their demands are to be fully met. We assume that the parallel transportation is done from all the sources but a source supplying to more than one destination serves them one after the other in any order. The total time used by the source is the sum of the shipment time for the destinations, which receive some or all the units of their requirement from the i^{th} source. The net time for the entire shipment is the maximum total time among all the sources. The decision variable x_{ij}, $(i,j) \in I \times J$ represents the units of good transported from the i^{th} source to the j^{th} destination. Mathematically Problem can be formulated as:

$$\left.\begin{array}{llll}
\min T(X) = \max\limits_{i \in I} \left(\sum\limits_{j \in J} (t_{ij} : x_{ij} > 0) \right) & & & \\
\text{subject to,} & & & \\
\quad \sum\limits_{j=1}^{n} x_{ij} = a_i & , \quad i \in I & \ldots & (1) \\
\quad \sum\limits_{i=1}^{m} x_{ij} = b_j & , \quad j \in J & \ldots & (2) \\
\quad x_{ij} \geq 0 & , \quad (i,j) \in I \times J & \ldots & (3)
\end{array}\right\} \text{(P)}$$

Clearly for feasibility of the problem (P) we must have $\sum\limits_{i \in I} a_i = \sum\limits_{i \in J} b_j$. A solution is called a feasible solution if it satisfied *(1), (2), (3).*

Various authors [1, 2, 4, 6, 8, 12, 15] have studied the problems where the objective function seems similar to that of (P). In [8] a combinatorial problem is studied which deals with the minimization of the sum of a finite number of concave bottleneck functions subject to linear constraints. A related bottleneck linear programming problem is constructed whose k^{th} $(k \geq 1)$ extreme point solution satisfying certain conditions is proved to yield an optimal feasible solution. The objective function is different from $T(X)$ defined in (P). Categorized bottleneck assignment problem (BAP) have been studied in literature in [1, 2, 14, 15] where, the set of n jobs is partitioned into m categories $1<m<n$. These authors have tackled two separate cases of this problem. In the first case, jobs in each category are performed in sequential order but the jobs in various categories can be commenced simultaneously, whereas in the other case jobs within each category are commenced simultaneously but the k^{th} category is commenced only when all the jobs in the category $(k-1)$ have been completed. Seashan [15], Aggarwal et al. [1] also discussed the real life examples of these two problems. A min-max linear programming problem with linear constraints is studied in [4,6] where, a partition of the index set of decision variables is considered. A travelling salesman problem under categorization is presented in [13]. Punnen et al. [12] studied generalized bottleneck and constrained bottleneck problem which assumes various combinatorial forms depending upon the nature of the finite set and its partition. They also found that even if in the generalized bottleneck transportation problem (GBTP) finite set and the family of subsets of this set does not have the combinatorial structure as that of generalized bottleneck problem (GBTP), it can be reduced to the form of (GBP) as the optimal solution of (GBTP) is attained at one of the extreme points.

Unlike in the problem (P) in the above mentioned problems the groups into which the sources or the destinations are partitioned are known and if these groups are not known their methodologies can't be applied. This is the main difference between the problem (P) and the above mentioned problems. It may be noticed that for any feasible solution of (P) there bound to be groups of the destinations being served by a source as it may supply to more than one destination, but these groups are not known before hand. Our aim is to find that feasible solution of (P) in which destinations are grouped in such a manner that the corresponding value of $T(X)$ is the minimum.

To the best of authors' knowledge no study on this type of time transportation problem has yet appeared in the literature and this itself is the main motivating force for the present study.

In order to meet our objective an algorithm is proposed based on the lexi-search approach [3, 10, 11]. The algorithm starts with an initial upper bound which is obtained using a heuristic approach and it also serves as a supplementary aid in reducing the search. Proposed algorithm examines the optimality of the initial upper bound and if found non-optimal it will help to generate the optimal transportation schedule at an early stage of the search by avoiding the examinations of all possible schedules and investigating only potential basic feasible solutions.

In the next section some definitions are given and some results are established based upon which, an algorithm is proposed in the section 3. Section 4 contains a numerical illustration and the last section contains an extension and some concluding remarks on the study.

2. THEORETICAL DEVELOPMENT

Theorem 1 *The objective function* $T(X) = \max_{i \in I} = \left(\sum_{j \in J} (t_{ij} : x_{ij} > 0 \right)$ *is a concave function.*

Proof It may be noted that the functions $T_i(x) = \sum_{j \in J} (t_{ij} : x_{ij} > 0),\ i = 1, 2, \ldots m$ are concave functions being the sum of finite number of concave functions, [5, 7, 9]

Let $X^1 = \{x_{ij}^1\}$ *and* $X^2 = \{x_{ij}^2\}$ be any two feasible solutions,

$$T(X^1) = \max_{i \in I} \left(\sum_{j \in J} (t_{ij} : x_{ij}^1 > 0) \right)$$

and $T(X^2) = \max_{i \in I} \left(\sum_{j \in J} (t_{ij} : x_{ij}^2 > 0) \right)$

Let $X = \lambda X^1 + (1-\lambda) X^2,\ 0 \le \lambda \le 1$

(i) $\lambda = \mathbf{0}$ $X = X^2 \Rightarrow T(X) = T(X^2)$

$\Rightarrow$ $T(\lambda X_1 + (1-\lambda) X_2) = \lambda T(X_1) + (1-\lambda) T(X^2)$

(ii) $\lambda = \mathbf{1}$ $X = X^1 \Rightarrow T(\lambda X_1 + (1-\lambda) X_2) = T(X^1) = \lambda T(X^1) + (1-\lambda) T(X^2)$

(iii) $\boldsymbol{0 < \lambda < 1}$ If the $(i,j)^{th}$ component of X^1, say $x_{ij}^{\ 1}$, is positive then $(i,j)^{th}$ component of X is also positive, similarly if $(i,j)^{th}$ component of X^2 is positive then $(i,j)^{th}$ component of X is also positive. Therefore,

$T(X) \geq T(X^1)$ and $T(X) \geq T(X^2)$, and hence

$T(X) \geq \lambda\, T(X^1) + (1-\lambda)\, T(X^2),\ 0 < \lambda < 1.$

Therefore, in all the cases we have

$T(\lambda X^1 + (1-\lambda) X^2) \geq \lambda T(X^1) + (1-\lambda) T(X^2),\ \lambda \in [0, 1]$

Hence $T(X) = \max_{i \in I} \left(\sum_{j \in J} (t_{ij} : x_{ij} > 0) \right)$ is a concave function.

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Corollary The optimal solution (P) is a basic feasible solution [7, 9].

The algorithm proposed in the next section generates and examines only the appropriate **basic feasible solutions.**

Any basic feasible solution of the problem (P) can also be represented as:

$$w = ((j_1^1, j_2^1, \ldots j_{\ell 1}^1), (j_1^2, j_2^2, \ldots, j_{\ell 2}^2), \ldots, (j_1^m, j_2^m, \ldots, j_{\ell m}^m)) \qquad (4)$$

where, $\ell_i \geq 1,\ i \in I,\ \sum_{i \in I} \ell_i \leq m + n - 1$, and the i^{th} element $(j_1^i, j_2^i, \ldots j_{\ell i}^m)$ in w represents ℓ_i destinations being served by the i^{th} source. Note that for some $t \in \{1, 2, \ldots, \ell_i\}$ and some $s \in \{1, 2, \ldots, \ell_r\}$ we may have $j_t^i = j_s^r$ but $j_p^i \neq j_q^i$ for $p, q = 1, 2, \ldots, \ell_i$. Clearly $|w| = |I| = m$.

Each basic feasible solution in the form (4) can be though of as a sword of length m whose each element represents the destinations being served by a source. Also for each $i \in I$

$$x_{i j_t^i} > 0 \text{ for } t = 1, 2, \ldots, \ell_i.$$

$$x_{ij} = 0\ ,\ j \in J / \{j_1^i, j_2^i, \ldots, j_{\ell i}^i\}$$

The value of the objective function corresponding to w is denoted by $T(X^w)$. It is given by:

$$T(X^w) = \max_{i \in I} \left(\sum_{t=1}^{\ell_i} (t_{ij_t}^{i}) \right)$$

Clearly, improvement in $T(X^w)$ can only be obtained by interchanging or replacing some of the entries of the set J occurring in some elements of the word w.

Notations

T_0 : starting upper bound on the optimal value of the objective function $T(.)$

J' : index set of destinations whose demands are not yet fully met

T_u : updated upper bound on the value of the objective function

b_j^u : updated demand of the j^{th} destination

$(j \in J', \text{ if } b_j^u \neq 0$

a_i^u : updated availability at the i^{th} source

AB : Alphabet Matrix (defined on next page)

K_i : position of an entry in the i^{th} row of AB corresponding to which the destination $ab(i, K_i)$ is chosen first to be served by the i^{th} source.

L_i : position of an entry other than K_i in the i^{th} row of AB currently under study ($K_i = L_i$ only when $a_i \leq b^u{}_{iab(i,K_i)}$)

$\uplus$: augmentation, $\not\uplus$: negation of augmentation

Definitions

Alphabet Matrix (AB)

It is $m \times n$ matrix formed by the positions of the elements of the given $m \times n$ matrix $\{t_{ij}\}$ of time. The i^{th} row of AB consists of the positions of the entries in the i^{th} row of the matrix $\{t_{ij}\}$ when these are arranged in the non-decreasing order of their values. The y^{th} entry in the i^{th} row of AB represented by $ab(i, y)$. Therefore, $ab(i, 1) \in J$

represents the destination that may be served partially or wholly in the least possible time by the i^{th} source. That is, $\min_{j \in J} (t_{ij}) = t_{iab(i,\,1)}$. If $y < z$, then $t_{iab(i,\,y)} \le t_{iab\,(i,\,z)}$. Thus the i^{th} row of AB is : $[ab(i, 1), ab\,(i, 2), \ldots\ldots, ab(i, n)]$ and clearly $t_{iab(i,\,1)} \le t_{iab(i,\,2)} \le \ldots \le t_{iab(i,n)}$.

Partial word *(Pw)*

$$Pw = ((j_1^1, j_2^1, \ldots, j_{\ell 1}^1), (j_1^2, j_2^2, \ldots, j_{\ell 2}^2), \ldots, (j_1^r, j_2^r, \ldots, j_{\ell r}^r)), r \le m,$$

$$= \left(ab(1, y_1^1), ab(1, y_2^1), \ldots, ab(1, y_{\ell 1}^1)\right), \ldots (ab(r, y_l^2), \ldots, ab(r, y_{\ell r}^r)) \qquad (5)$$

$= Pw^1 \uplus Pw^2 \ldots\ldots \uplus Pw^r$, represents a partial word

where, $Pw^i = (j_1^i, j_2^i, \ldots\ldots j_{\ell 1}^i) = (ab(i, y_1^i), ab(i, y_2^i), \ldots\ldots\ldots, ab(i, y_{\ell i}^i))$, $i = 1, 2, \ldots, r$ is the i^{th} element of the partial word and represents the destinations being served by the i^{th} source. Partial solution corresponding to the partial word Pw is denoted by X^{Pw} and it consists of supplying the destinations $j_1^i, j_2^i, \ldots\ldots, j_{l_i}^i$ by the i^{th} source, $i = 1, 2, \ldots, r$

(sources $r + 1, r + 2, \ldots\ldots, m$ are still to be studied).

Partial word Pw defines a block of words each of which has first r elements as $Pw^1, Pw^2, \ldots\ldots, Pw^r$. In this sense Pw is called the leader of this block of words. Contribution to the objective function $T(.)$ by the partial solution , say X^{Pw}, corresponding to Pw is given by

$$T(X^{Pw}) = \max_{i \in \{1,\ 2, \ldots, r\}} (T(X^{P_w^i}))$$

Length of the i^{th} element of a partial word is denoted by $\left|Pw^i\right|$ and that of partial word by $\left|Pw\right|$. From (5), $\left|Pw^i\right| = \ell_i, i, = 1, 2, \ldots, r$ and $\left|Pw\right| = r$. A partial word of length m is called a word. Thus for a word, say w, whose leader is Pw, $T(X^{Pw}) \le T(X^w)$. A partial word of length r is generated systematically by considering the entries of the first r rows of AB. These are generated in decreasing order of their contribution to the objective function $T(.)$.

The first entry j_1^i in Pw^i represents the destination chosen first to be served by the i^{th} source. In case a_i is fully utilized while serving destination j_1^i , then $Pw^i = (j_1^i)$ and if some units of a_i are still available after serving the destination j_1^i then, a destination j_2^i is chosen and so on until $a_i^u = 0$ and then the same procedure is repeated for the next source and so on till the availability of the m^{th} source is exhausted i.e., till $a_m^u = 0$.

Suppose current upper bound on the value of $T(.)$ is T_u. Le the current partial word obtained after studying the first r sources be $Pw = Pw^1 \uplus Pw2 \uplus ... \uplus Pw^r$. As partial words are generated systematically in non-decreasing order of their contribution to the objective function $T(.)$, it follows that none of the partial words having the first entry in their r^{th} element as $ab(r, z_1^r), 1 \leq z_1^r < y_1^r$ can yield value of the objective function less than T_u, where $Pw^r = \left(ab(r, y_1^r), ab(r, y_2^r), ..., ab(r, y_{l_r}^r)\right)$.

Theorem 2 *Let*

$$\overline{Pw} = \left(ab(1, y_1^1), ab(1, y_2^1), ..., ab(1, y_{l_1}^1)\right), ...\left(ab(r-1, y_1^{r-1}), ab(r-1, y_2^{r-1}) ....., abr-1, y_{1_{r-1}}^{r-1}\right)$$

$= Pw^1 \uplus Pw^2 \uplus .... \uplus Pw^{r-1}$ be a partial word of length r – 1 and the rth source be under study i.e., $Pw^r = \left(ab(r, y_1^r), ab(r, y_2^r), ..., ab(r, y_{l_r}^r)\right)$ and $a_r^u \neq 0$. If for all

$y_1^r = 1, 2, ..., n$ $T(X^{Pw^r}) \geq T_u$, the $\overline{P}w$ can't contain a word with corresponding value of the objective function T(.) less than T_u.

Proof As for all $y_1^r = 1, 2, .., n$ $T(X^{P_w^r}) \geq T_u$, it follows that the destination which is chosen first to be served by the r^{th} source yields a Pw^r for which the contribution to the objective function is no better than T_u. This implies that the partial word $\overline{Pw}$ can't contain a word, say w, corresponding to which the associated solution yield value of $T(.)$ less than T_u.

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Remark 1 Generation of $\mathbf{Pw^r}$ Let $\overline{Pw} = Pw^1 \uplus Pw^2 \uplus ..... \uplus Pw^{r-1}$, $r < m$. Clearly i^{th} source $(i = 1, 2, ... r- 1)$ is supplying goods to the destinations $ab(i, (i, y_1^i), ..., ab(i, y_{\ell i}^i)$ which constitute Pw^i and $a_i^u = 0$.

Initially when the r^{th} source is picked up for study $Pw^r = \phi$. Set $K_r = 1$. If $ab(r, K_r) \in J'$, then set $x_{rab(r,\ Kr)} = min\ (a_r, b^u_{ab(r,K_r)})$ and update the availability of the r^{th} source as $a_r - x_{rab(r,\ Kr)} = a^u_r$ and requirement of the destination $ab(r, K_r)$ as $b^u_{ab(r,K_r)} - x_{rab(r,K_r)}$. If $ab(r, Kr) \notin J'$, set $K_r = K_r + 1$ and continue likewise until a destination $ab(r, K_r) \notin J'$ is obtained with $t_{rab(r,\ Kr)} < T_u$ and update as explained above. If updated availability a^u_r after fixing $x_{rab(r,\ Kr)}$ is zero then Pw^r is updated as $Pw^r \uplus (ab(r, K_r))$ and $\overline{\text{Pw}}$ is updated as $\overline{\text{Pw}} \uplus \text{Pw}^r$. If a^u_r after fixing $x_{rab(r,\ Kr)}$ is positive, then updated Pw^r is $Pw^r \uplus (ab(r, K_r))$, $J' = J'/(ab(r\ K_r))$ and set $L_r = t'$ where t' is such that $t_{rab(r,t')} = \min_{ab(r,s) \in J'} (t_{rab(r,\ s)})$, and $b^u_{ab(r,\ t')} \geq a^u_r$. Set $x_{rab(r,\ Lr)} = a^u_r$ and update $a^u_r, b^u_{ab(r,\ t')}$ as before and $Pw^r = Pw^r \uplus (ab(r,L_r))$. Now $a^u_r = 0$ and if $T\left(X^{Pw^r}\right) \leq T_u$, then Pw^r is not augmented to $\overline{\text{Pw}}$ and Pw^r is altered. If no such t' exist , then set $L_r = 1$ if $K_r \neq 1$ or $L_r = 2$ if $K_r = 1$. If $ab(r, L_r) \in J'$, then set $J' = J'/\{ab(r,\ L_r)\}, x_{rab(r,L_r)} = b^u_{ab(r,\ L_r)}$ and updated a^u_r and $b^u_{ab(r,L_r)}$ as before. Update $Pw^r = Pw^r \uplus (ab(r,\ L_r))$ and since a^u_r is still positive set $L_r = t''$ where t'' such that $t_{rab(r,\ t'')} = \min_{ab(r,s) \in J'} (t_{rab(r,s)})$ and $b^u_{ab(r,\ t'')} \geq a^u_r$ and update as in case of t' but if no such t'' exist then set $L_r = L_r + 1$ and continue allocating in the r^{th} source until $a^u_r = 0$ and update as explained.

Notice that if for all changes of Pw^r we have $T(X^{Pw^r}) \geq T_u$, then $\overline{Pw}$ cannot contain a word with the corresponding solution X^w yielding value of $T(.)$ less than T_u. In such a case $\overline{Pw}$ has to undergo "alternation".

Remark 2 If $T\ (X^{Pw}) < T_u$, then the partial word Pw may contain a word with the corresponding value of $T(.)$ less than T_u. And if $T(X^{Pw}) \geq T_u$, then Pw is abandoned and undergoes "alteration".

Remark 3 Let

$\overline{Pw} =$

$\left(ab(1,y_1^1), ab(1, y_2^1),..(ab(1, y_{l_1}^1)),...(ab(r-1,y_1^{r-1})), ab(r-1,y_2^{r-1}),..,ab(r-1,y_{lr-1}^{r-1})\right)$

$r \le m$ and $Pw^r = \phi$. If $t_{rab(r,\, Kr)} \ge T_u$ and Pw^r is augmented with $ab(r, K_r)$, then $T(X^{Pw^r})$ $\ge T_u$, and as such $\overline{Pw}$ can't contain a word, say w, with corresponding value of $T(.)$ less than T_u. This is so because $t_{rab(r,\, z)} \ge T_u$, for all $z \ge K_r + 1$ due to the nature of the Alphabet matrix and $ab(r, y) \notin J'$ for all $y \le K_r - 1$ as per the definition of K_r. Hence $\overline{Pw}$ must undergo alteration if $t_{rab(r,\, Kr)} \ge T_u$.

Remark 4 Let $\overline{\mathrm{Pw}}$ be a partial word of length (r – 1) with $T\left(X^{\overline{Pw}}\right) < T_u$. Let $Pw^r = \left(ab(r, y_1^r),...,(ab(r, y_t^r))\right)$ and $a_r^u > 0$. Let $T(X^{Pw^r}) \ge T_u$. The further augmentation in Pw^r is not advisable and as such Pw^r must be altered. If Pw^r is such that $K_r = y_1^r = n$ and $T(X^{Pw^r}) \ge T_u$, then again $\overline{Pw}$ must undergo alterations.

Remark 5 (Alteration in a 'elelment' and Partial word)

a. Alteration in an element, say $\mathbf{Pw^r}$

Let $\overline{Pw} = Pw^1 \uplus Pw^2 \uplus ,.... \uplus P^{r-1}$ where $Pw^i = \left((ab(i, y_1^i),(ab(i,y_2^i),.....ab(i,y_{l_i}^i))\right)$. Let partial generation of Pw^r yield $Pw^r = \left((ab(i, y_1^i),(ab(i,y_2^i),.....(ab(i,y_{l_i}^i))\right) > 0$. If this Pw^r is such that $T(X^{Pw^r}) \ge T_u$, then Pw^r must under go alterations. (Ref. Remark 4).

If $K_r = y_1^r < n$ then set $Pw^r = \phi$ and start regenerating Pw^r with K_r, replaced by $K_r + 1$ after updating $a_r^u = a_r, b^u_{ab(r,y_s^r)} = x_{rab(r,y_s^r)}$, $s = 1, 2,...., t$. If $K_r = y_1^r = n$ then set $Pw^r = \phi$,

$a_r^u = a_r, b^u_{ab\left(r, y_S^r\right)}$ = $a_r^u = a_r, b^u_{ab\left(r, y_s^r\right)}$, s = $1, 2,\ldots,t$. If K_r = y_1^r = n then set Pw^r = ϕ,

$a_r^u = a_r, b^u_{ab\left(r, y_s^r\right)}$ = s = $1, 2, \ldots..t$, and alter $\overline{Pw}$.

If Pw^r = ϕ and $t_{rab(r,\, Kr)} \geq T_u$, then again $\overline{Pw}$ undergoes alteration (Ref. Remark 3).

b. Alteration in $\overline{Pw}$

Set $\overline{Pw}$ = $\overline{Pw} \ominus$ Pw^{r-1}. Pw^{r-1} is s altered as Pw^{r-1} = $Pw^{r-1} \ominus$ $ab\left((ab\,(r-1, y^{r-1}_{l_{r-1}})\right)$, set $J' = J' \cup (ab\left(r-1\right), y^{r-1}_{l_{r-1}}))$ if $ab(r-1), y^{r-1}_{l_{r-1}}) \notin J'$, set

$a^u_{r-1} = x_{r-lab(r y^{r-1}_{\ell_{r-1}})} = b^u_{ab(r-1, y^{r-1}_{\ell_{r-1}})} = b^u_{ab(r-1, y^{r-1}_{\ell_{r-1}})} + x_{ab(r-1, y^{r-1}_{\ell_{r-1}})}$, and L_{r-1} is set equal to

t' where $t_{r-1ab(r-1,t')} = \min\limits_{j \in J'} t_{r-1j}$ and $b^u_{ab(r-1,t')} \geq a^u_{r-1}$. If such a t' does not exist, then set L_{r-} = $y^{r-1}_{l_{r-1}}$ + 1 and augment $ab(r\text{-}\ 1,\ L_{r-1})$ to Pw^{r-1} and update $a^u_{r-1,}\, b^u_{(abr-1, L_{r-1})}, J'$ etc. as explained in Remark 1. If Pw^{r-1} can't be altered to generate a better partial word, then set Pw^{r-1} = ϕ, $a^u_{r-1} = a_{r-1}, b^u_{ab(r-1, y_S)} = x_{r-1ab(\,r-1, y_S\,)}$, s = $1, 2, \ldots..,\ \ell_{r-1}$, negate Pw^{r-2} from $\overline{Pw}$ i.e., set $\overline{Pw}$ = $\overline{Pw} \ominus Pw^{r-2}$ and now alter P_w^{r-2} as explained in (a).

Theorem 3 *T_u is the current upper bound on the value of T(.). Let Pw = ϕ. T_u is the optimal value of T(.) if any one of the following holds.*

i. Pw^1 = ϕ *and* $t_{1ab(1,\ K_1)} \geq T_u$.

ii. Pw^1 = $ab(1, y_1^1), ab\ (1, y_2^1), \ldots.., ab(1,\ y^1_{\ell_1}), a_1^u \geq 0, T(X^{Pw^1}) \geq T_u; y_1^1 = K_1 = n.$

iii. Pw^1 = $ab\left(1, y_1^1\right), \ldots\ldots, ab(\,1, y^1_{\ell_1}\,), a_1^u \geq 0, T(\,X^{Pw^1}\,) < T_u;\ y_1^1 = K_1 = n$ *and* $L_1 = n\text{-}1$.

Proof

i. As $t_{1ab(1,\ K_1)} \geq T_u$ and Pw^1 = ϕ. Pw must undergo alteration (Ref. Remark 3). But as $Pw = \phi$, no alteration is possible.

ii. As $y_1^1 = K_1 = n$ and $T(X^{P_w^1}) \geq T_u$, Pw must undergo alteration in accordance with Remark 4. Since $Pw = \phi$ no alteration is possible and therefore no new better word can be generated.

iii. As $T(X^{Pw^1}) < T_u$, Pw^1 may generate an element of Pw after augmentation or alteration. On augmenting this new Pw^1 to Pw we got a partial word with a better value of $T(.)$. But as $y_{l_1}^1 = L_1 = n = n - 1$ augmentation in the current Pw^1 is not possible and as $y_1^1 = K_1 = n$ regeneration of Pw^1 is also not possible.

Also in all the above three cases as the partial words are generated in a systematic manner, it follows that no partial word $Pw = Pw^1$ with $ab(1, y)$, $y < K_1$ can generate a word with the corresponding value of the objective function $T(.)$ less then T_u. Hence, T_u is the optimal value of $T(.)$.

A Heuristic to obtain the initial upper bound on the optimal value of the objective function:

Initially $I_u = I$, $J' = J$ where, I_u is the index set of those sources for which $a_i^u \neq 0$. Let t stands for the number of the iterations. For $t = 1, 2, ..., m-1$, find $\max_{i \in I_u} a_i^u = a_{i_t}^u$ (say). Find penalties corresponding to each $j \in J'$ (difference of the minimum from the next minimum entry in that column). Choose that $j \in J'$ corresponding to which penalty is greatest. Let it be j'. Set $x_{i_t j'} = \min\left(a_{i_t}^u, b_j^u\right)$ and update $a_{i_t}^u = a_{i_t}^u - x_{i_t j'}, b_{j'}^u = b_{j'}^u - x_{i_t j'}$. If $a_{i_t}^u = 0$, then $I_u = I_u \backslash \{i_t\}$ and if $b_{j'}^u = 0$, then $J' = J' \backslash \{ j' \}$. If $a_{i_t}^u \neq 0$, then choose j'' such that $\min_{j \in J'} t_{i_t j''} = t_{i_t j''}$ and update as for the case of j' and continue likewise until $a_{i_t}^u = 0$.

For $t = m$, let $I_u = \{i\}$. Set $x_{ij} = b_{j'}^u$ $\forall j \in J'$ and set $I_u = \phi$, $J' = \phi$. We have obtained a basic feasible solution $X = \{x_{ij}\}$ of the problem (P) and the corresponding value of the objective function provides the initial upper bound on the optimal value of $T(.)$. It is denoted by T_0.

3. Algorithm

Through out the algorithm following we update as follows :

(U-1) Augmentation in P_w^r

Suppose $Pw = \left(ab(1,y_1^1),...,ab(1,y_{l_1}^1)\right),......,\left(ab(r-1,y_1^{r-1}),......,ab(r-1,y_{\ell_{r-1}}^{r-1})\right)$

$= \left((j_1^1, j_2^1,...,j_{l_1}^1),........,(j_1^{r-1}, j_2^{r-1},...,j_{l_{r-1}}^{r-1})\right)$

$= \quad Pw^1 \uplus Pw^2, \uplus .. \uplus Pw^{r-1}.$

and the *rth* source is being studied i.e., the r^{th} element Pw^r has been partially generated which means $a_r^u > 0$, then

a. if $\exists\, j \in J'$ such that $b_j^u \geq a_r^u$, then set $x_{rj} = a_r^u$. Then b_j^u, a_r^u, Pw^r are updated as $b_j^u - a_r^u$, 0, $Pw^r \uplus (j)$ respectively. If $b_j^u = a_r^u$, then $J' = J'/\{j\}$. Now the updated Pw^r is a fully generated element meaning thereby that the availability of the r^{th} source is fully exhausted.

b. if $\exists\, j \in J'$ such that $b_j^u < a_r^u$, then set $x_{rj} = b_j^u$ now b_j^u, a_r^u and Pw^r are updated as 0, $a_r^u - b_j^u$, $Pw^r \uplus (j)$ respectively. Updated Pw^r will still be a partially generated element i.e.

$a_r^u > 0 \ \ but \ \ J' = J' \setminus \{ j \}.$

(U-2) Negation of augmentation in Pw^r

Suppose $Pw = Pw^1 \uplus Pw^2 ... \uplus Pw^{r-1}$ where $Pw^i = (j_1^i, j_2^i,....,j_{l_i}^i)$, $i = 1, 2,....,r-1$ and

$P_w^r = (ab\left(ab(r,y_1^r), ab(r,y_2^r),.....,ab(r,y_s^r)\right) = (j_1^r, j_2^r,....,j_s^r)$ is such that $a_r^u \geq 0$

(a). If each of Pw^r is to be taken out for alteration, then set

$Pw^r = \phi$, $a_r^u = a_r$, $J' = J' \bigcup \{j_1^r, j_2^r,......, j_s^r\}$, $b_{j_t^r}^u = b_{j_t^r}^u + x_{rj_t^r}$, $t = 1, 2, ..., s$. Set

$x_{rj_t^r} = 0, \forall\, t = 1,2,..,s$

(b). If only last component of Pw^r is to be taken out for alteration, then update

$Pw^r = Pw^r \Cap (j_s^r)$, $b_{j_s^r}^u = b_{j_s^r}^u + x_{rj_s^r}$, $a_r^u = a_r^u + x_{rj_s^r}$, $J' = J' \bigcup \{j_s^r\}$, $x_{rj_s^r} = 0$

(U-3) Negation of augmentation in Pw

Let $Pw=Pw^1 \uplus Pw^2 \uplus ,..., \uplus Pw^s \uplus .. \uplus Pw^r$, where $Pw^i=(j_1^i, j_2^i,..., j_{\ell_i}^i)$, $i=1,2,\ldots$,r.

When last few elements of Pw are to be negated, then

set $Pw = Pw \ominus (Pw^s \uplus Pw^{s+1} \uplus ...., \uplus Pw^r)$

$= Pw^1 \uplus Pw^2 \uplus ..... \uplus Pw^{s-1}$,

$Pw^i = \phi$, $i = s + 1,\ldots\ldots, r$, $Pw^s = (j_1^s, j_2^s,....., j_{l_s}^s)$, $a_i^u = a_i$, $i = s + 1,...., r$

$$b_j^u = b_j^u + \sum_{i=s+1}^{r} x_{ij}, \; j \in \left\{ j_1^{s+1} ..., j_{l_{s+1}}^{s+1}; \, j_1^{s+2} ,...., j_{l_{s+2}}^{s+2} ;..., j_1^r ,..., j_{l_r}^r \right\}$$

$$J' = J' \cup \left\{ j_1^{s+1}, j_2^{s+2} ,..., j_{l_{s+1}}^{s+1} \right\} \cup \left\{ j_1^{s+2} ,..., j_2^{s+2} ,..., j_{l_{s+2}}^{s+2} \right\} \cup ... \cup \left\{ j_{1,}^r, j_2^r ,..., j_{l_r}^r \right\}.$$

The algorithm runs in the following steps:

Step 0 From the Alphabet matrix *AB* and find the initial upper bound T_0 on the optimal value of the objective function *T(.)*. Set $T_u = T_0$ and go to Step 1.

Step 1 Set $i = 1$, $K_i = 1$, $Pw = \phi$, $Pw^i = \phi$, $J' = J$. Go to Step 3A(α).

Step 2 a. Update as in (U-2) (a). Set $i = i - 1$, $Pw = Pw \ominus Pw^{i-1}$,

$Pw^i = \left(ab(i, y_1^i), ab(i, y_2^i),....,ab(i, y_{l_i}^i) \right)$, $K_i = y_1^i$, $L_i = y_{l_i}^i$ and got to Step 5.

b. Update as in (U-2) (a). Set $K_i = K_i + 1$, go to Step 3A (α).

c. Update as in (U-2) (b). For the same *i* and K_i go to Step 3A α(a).

Step 3 (A) When i ≠ m

(α) When ab (i, K$_i$) $\in J'$

Let $t_i\, ab(i, K_i) \geq T_u$. *If* $i = 1$, go to Step 8, and if $i > 1$ go to Step 2 (a).

Let $t_{iab(i, Ki)} < T_u$.

i. If $b_{ab(i,\,ki)}^u \geq a_i$ then update as in (U-1) (a). Set $Pw = Pw \uplus Pw^i$ and go to Step 6.

ii. If $b^u_{ab(i,\ ki)} < a_i$ then update as in (U-1) (b). If possible go to (a), otherwise set $L_i = 1$, *if* $K_i \neq 1$, or $L_i = 2$, if $K_i = 1$, and go to(b).

(a). Set $Li = t'$, where t' is such that $ab(i,\ t')$ is the first unstudied entry in the i^{th} row of AB for which $ab(i,\ t') \in J'$ and $b^u_{ab(i,t')} \geq a^u_i$. Update as in (U-1) (a).

if $\sum_{j \in Pw^i} t_{ij} < T_u$, then set $Pw = Pw \uplus Pw^i$ and go to Step 6.

if $\sum_{j \in Pw^i} t_{ij} \geq T_u$, then with the same Pw, Pw^i go to Step 4 (a).

(b) (i) If $ab(i, L_i) \in J'$ then update as in (U-1) (b). Further if if

$\sum_{j \in Pw^i} t_{ij} < T_u$ then go Step 3Aα (a) or Step 4(b) which ever is the case, and

if $\sum_{j \in Pw^i} t_{ij} \geq T_u$ then go to Step 4(a).

(ii) If $ab(i, L_i) \notin J'$ then go to Step 4(b).

(β) When ab(i, K_i) $\notin J'$

If $K_i = n$ or $ab(i,\ y) \notin J'$ for $y = K_i + 1, \ldots, n$, then go to Step 2(a), otherwise repeat Step 3(A) for $K_i = K_i + 1$.

B When *i = m*

Set $x_{ij} = b^u_j \ \forall j \in J'$

Now $Pw_i = Pw_i \uplus J', a^u_i = 0, b^u_j = 0 \forall j, J' = \phi$.

Find $\sum_{j \in Pw^i} t_{ij}$

if $\sum_{j \in Pw^i} t_{ij} < T_u$, *set* $Pw = Pw \uplus Pw^i$ and go to Step 6, and

if $\sum_{j \in Pw^i} t_{ij} \geq T_u$, go to Step 2(a)

Step 4 (a) If for $K_i = y^i_1$, $L_i = y^i_{li}$, y^i_1 is such that $ab(i, y) \notin J'$ for $y = y^i_1 + 1, \ldots, n$ or if $y^i_1 = n$ then go to Step 2(a), otherwise go to Step 2(b).

(b) If for $K_i = y_1^i, L_i = y_{\ell_i}^i, y_{\ell_i}^i$ is such that either $ab(i,y) \notin J'$ for $y = y_{\ell_i}^i + 1,...,n$ or $y_{\ell_i}^i = n$ or $/Pw^i/=1.$ then go to Step 2(b), otherwise go to Step 3Aα (b) for $L_i = L_i+1.$

Step 5 If $i = 1$ and any of the conditions mentioned in Theorem 3 are satisfied then go to Step 8. Otherwise, (i) go to Step 2(a) if $/Pw^i/ = 1, i \neq 1$ or (ii) go to Step 2(c) if $/Pw^i/>1.$

Step 6 (a) If $i < m$, then in virtue or Remark 2, set $i = i + 1, Pw^i = \phi$. If updated i is less than m, go to Step 3(A) for $K_i = 1$ and if updated $i = m$, go Step 3 (B).

(b) If $i = m$, we have found a new (better) upper bound on the optimal value of T(.), corresponding to which, let the new basic feasible solution be w. Set $Pw = w$, $T_u = T(X^w)$ and go to Step 7.

Step 7 Let $r \in I$ be such that $T_u = \max_{i \in I} \left(\sum_{t=1}^{l_i} t_{ij_t^i} \right) = \sum_{t=1}^{l_r} t_{rj_t^r}$ then update as in (U-3) by taking $r = s$, $m = r$, Set $i = r$,

$Pw^i = (ab(i,y_1^i), ab(i,y_2^i),...,ab(i,y_{l_i}^i)), K_i = y_1^i, L_i = y_{l_i}^i$,. and go to Step 4(a).

Step 8 Stop. The current upper bound T_u is the optimal value of $T(.)$ and the corresponding basic feasible solution is an optimal feasible solution.

Remark 6 While fixing x_{ij} at any stage we set $x_{ij} = \min\left(a_i^u, b_j^u\right)$, $j \in J'$ and delete the i^{th} row or the j^{th} column according as the minimum is a_i^u or b_j^u. This ensure that only basic feasible solution are generated and examined in the process [9].

4. Numerical Illustration

Consider the following time minimizing transportation problem having four sources $S_i, i=1, 2, 3, 4$ with respective availabilities $a_i = 40, 45, 50, 45$ and five destinations $D_j, j=1,2,..,5$, with demands $b_2 = 40, 50, 35, 30, 25$ respectively.

$\{t_{ij}\}$ =

	D1	**D_2**	**D_3**	**D_4**	**D_5**	**a_i**
S_1	**3**	4	2	**2**	5	40
S_2	4	1	**2**	4	**2**	45
S_3	3	**2**	4	5	3	50
S_4	**2**	5	1	3	4	45
b_j	40	50	35	30	25	

Step 0 Construct Alphabet matrix :

AB =

3	4	1	2	5
2	3	5	1	4
2	1	5	3	4
3	1	4	5	2

Also find the initial upper bound on the value of *T(.)* as explained in Section 2 corresponding allocations are given in the following table:

	10	0	0	30	0
$X =$	0	0	20	0	25
	0	$x_{32}=50$	0	0	0
	30	0	15	0	0

This solution can be represented by $w = ((4, 1), (3, 5), (2), (1, 3))$, $T(X^w) = 5$, Set $T_u = T_0 = 5$ and go to Step 1.

Step 1 Set $i = 1, K_i = 1, Pw = \phi, Pw^i = \phi, J' = J$. Go to Step 3A (α)

Step 3 (A) $ab(1, K_1) = 3 \in J'$, $t_{13} = 2 < T_u$ ($=5$), $b_3 = 35 < a_1$ ($= 40$)

(α) Set $x_{13} = 35$, $b_3^u = 0$, $a_1^u = 5$, $J' = J' \setminus \{3\} = \{1, 2, 4, 5\}$, $Pw^1 = Pw^1 \uplus (3) = (3)$.

(a) Set $L_1 = 2$, $x_{1ab(1, 2)} = x_{14} = 5$, $b_4^u = 30 - 5 = 25$, $a_1^u = 0$, $Pw^1 = Pw^1 \uplus (4) = (3,4)$.

$\sum_{j \in Pw^1} t_{ij} = 4 < T_u$ $(= 5)$. Go to Step 6.

Step 6 (a) $i < m$, set $i = i+1 = 2$, $Pw^2 = \phi$, updated i $(=2) < m$. Set $K_2 = 1$ and go to Step 3 (A)

Step 3(A) $ab(2, 1) = 2 \in J'$, $t_{2ab(2, 1)} = t_{22} = 1 < T_u$ $(= 5)$,

$b_2 = 50 > a_2(= 45)$, $Pw = (3, 4) \uplus (2) = ((3, 4), (2))$, $b_2^u = 5$, $a_2^u = 0$, $J' = \{1, 2, 4, 5\}$. Go to Step 6.

Proceeding like this we will generate a word $Pw = w = ((1), (5, 3), (2), (3, 4))$, $T(X^{Pw}) = 4 < T_u$. Thus we get a better upper bound on the value of the objective function.

Investigating this word for optimality leads to a step where $i = 1$, $Pw = \phi$, $Pw^1 = \phi$, $K_1 = 4$, $t_{1ab(1,4)} = t_{12} = 4 = T_u$. The process ends in virtue of Theorem 3. Hence, optimal solution is yielded by the word *((1), (5, 3), (2), (3, 4))* and the optimal value of *T(.)* is 4.

5. Concluding Remarks

Imbalanced Time Minimizing Transportation Problem

In case of Imbalanced Time Minimizing Transportation Problem (TMTP) we have $\sum_{i\in I} a_i \neq \sum_{j\in J} b_j$.

Let $\sum_{i\in I} a_i > \sum_{j\in J} b_j$. Then mathematical formulation of the problems is:

$$\min\ (T(X) = \max_{i\in J} (\sum_{j\in J} t_{ij} : x_{ij} > 0)$$

subject to,

$$\sum_{j=1}^{n} x_{ij} \leq a_i \quad , \ i \in I$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ , \quad j \in J$$

$$x_{ij} \geq 0 \quad , \quad \forall (i,j) \in I \times J$$

The equivalent standard (TMTP) whose optimal solution will provide an optimal solution of the above problem is:

$$\min T(X) = \max_{i \in J} \left(\sum_{j \in J'} t_{ij} : x_{ij} > 0 \right)$$

subject to

$$\sum_{j=1}^{n+1} x_{ij} = a_i \quad , \; i \in I$$

$$\sum_{i=1}^{m} x_{ij} = b_j \; , \; \; j \in J'$$

$$x_{ij} \; \geq \; 0 \quad , \quad \forall (i, j) \in I \times J'$$

where $J' = J \bigcup \{ n+1 \}$

$$t_{in+1} = 0, b_{n+1} = \sum_{i \in I} a_i - \sum_{j \in J} b_j \; .$$

The case $\sum_{i \in I} a_i < \sum_{j \in J} b_j$ can be similarly discussed.

Thus the proposed solution methodology can be applied to imbalanced TMTP's as well.

It has been shown here that the optimal solution of the problem (P) is attainable at an extreme point of the feasible solutions for obtaining the global optimal solution. However, in the proposed algorithm only a small subset of basic feasible solutions is examined by applying some sort of dominance check to reject a large number of extreme points for which the corresponding value of the objective function is not less then the current upper bound, Here, the algorithm converges in a finite number of steps. As a matter of fact, the simplicity and ease of computation of the developed algorithm make it possible to solve reasonably large sized problems.

In the absence of any other solution strategy, we are unable to present any comparative study. Infact, the main motivating force for the present study was to evolve a solution strategy for problem.

Acknowledgement

This research work is partially supported by the Council of Scientific and Industrial Research, India.

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