



# Ranking in quadratic integer programming problems

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## Abstract

The present paper develops an algorithm for ranking the integer feasible solutions of a quadratic integer programming (QIP) problem. A linear integer programming (LIP) problem is constructed which provides bounds on the values of the objective function of the quadratic problem. The integer feasible solutions of this related linear programming problem are systematically scanned to rank the integer feasible solutions of the quadratic problem in non-decreasing order of the objective function values. The ranking in the QIP problem is useful in solving a nonlinear integer programming problem in which some other complicated nonlinear restrictions are imposed which cannot be included in the simple linear constraints of QIP, the objective function being still quadratic.

*Keywords:* Integer programming; Quadratic programming; Linear programming

## 1. Introduction

In this paper a solution strategy is proposed for solving a programming problem of the following type:

(P)

$$\text{Min } f(X) = CX + X^TDX$$

subject to

$$AX = b,$$

$X \geq 0$  and an integer vector,

and  $X$  satisfies the additional complicated nonlinear restrictions  $h(X) \leq 0$ . Here,  $X \in \mathbb{R}^n$ ,  $C^T \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ ,  $D \in \mathbb{R}^{n \times n}$  is a symmetric real matrix and  $h$  is a  $p$ -dimensional nonlinear vector function.

A quadratic integer programming (QIP) problem closely related with problem (P) is the following:

(QIP)

$$\text{Min } f(X) = CX + X^TDX$$

subject to

$$AX = b,$$

$X \geq 0$  and an integer vector,

where it is assumed that the feasible region of QIP is nonempty and bounded.

Quadratic integer programming problems have wide application: in finance, as studied by Findlay [3], Lintner [11] and Markowitz [13]; in capital budgeting, as discussed by Weingartner [21], Bernhard [1], Mao [12], Laughhunn [10], Peterson [16] and Rathakrishnan [17]; and in scheduling, as described by Moder [15]. The solution procedures for obtaining an optimal solution of QIP are given by

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many authors, viz. Hammer [5,6], Glover [4], Hansen [7], McBride [14], Carter [2], Williams [22] and Kalantari [8,9].

Algorithms for ranking the integer feasible solutions in linear programming problems and linear fractional programming problems have been developed by Verma et al. [19,20]. To the best of the authors' knowledge ranking the integer feasible solutions of a nonlinear programming problem and in particular of QIP has not yet been taken up.

In this paper, to solve problem (P) an algorithm is developed to rank the integer feasible solutions of the QIP problem. If the  $k$ -th best ( $k \geq 1$ ) integer feasible solution of QIP is the first one to satisfy the additional complicated nonlinear constraints  $h(X) \leq 0$ , then that integer feasible solution will be the best (optimal) integer solution of problem (P).

If the additional constraints are linear, then they can be incorporated in  $AX = b$  and problem (P) becomes a QIP problem. In realistic situations the additional nonlinear constraints  $h(X) \leq 0$  may depict financial, time, social and other restrictions. This proposed method of ranking the integer feasible solutions may also be useful in bicriterion quadratic integer programming problems.

Section 2 of the paper deals with the theoretical development based upon which an algorithm for ranking in QIP is proposed in Section 3. A numerical illustration in support of the theory is included in Section 4.

## 2. Theoretical development

Let  $S$  be the set of feasible solutions of QIP. That is,

$$S = \{X \in \mathbb{R}^n : AX = b, X \geq 0 \text{ and an integer vector}\}.$$

The bounding linear integer programming (LIP) problem, which provides lower bounds on the objective function values of the QIP problem, is as follows:

(LIP)

$$\text{Min}_{X \in S} g(X) = (C + U)X,$$

where

$$U_j = j\text{-th component of } U^T \in \mathbb{R}^n$$

$$= \min_{X \in S} X^T D_j \quad (j = 1, \dots, n),$$

$D_j$  being the  $j$ -th column of  $D$ .

It may be noticed that since

$$U_j \leq X^T D_j \quad \forall X \in S, \quad j = 1, 2, \dots, n,$$

$$(C + U)X \leq CX + X^T DX, \quad X \in S.$$

Hence

$$g(X) \leq f(X) \quad \forall X \in S. \tag{1}$$

### Notations

$g_i = g(X_{i_j})$ ,  $x_{i_j} \in X_{i_j}$ , the set of the  $i$ -th best integer feasible solutions of LIP (the  $i$ -th best integer feasible solutions of LIP can be obtained as explained by Verma et al. [19]); obviously  $i = 1, 2, \dots, N$ , where  $g_N = \max\{g(X) : X \in S\}$

$T^r = \cup_{i=1}^r X_{i_j}$ ,  $r = 1, 2, \dots, N$ .

$f_k$  = The  $k$ -th best objective function value in QIP.

$L_k$  = The set of the  $k$ -th best integer feasible solutions of QIP.

Proposition 2.1, proved below, explains how an optimal integer feasible solution of QIP is obtained from LIP.

### Proposition 2.1. If

$$g_k \geq \min\{f(X) : X \in T^k\} = f(\hat{X}),$$

say, then  $\hat{X}$  is an optimal solution of QIP.

### Proof.

$$f(\hat{X}) = \min\{f(X) : X \in T^k\}$$

$$\Rightarrow f(X) \geq f(\hat{X}) \quad \forall X \in T^k. \tag{2}$$

As  $g_k$  is the value of  $g(X)$  at the  $k$ -th best integer feasible solutions of LIP,

$$g_w > g_k \quad \forall w \geq k + 1.$$

Also,

$$f(X_{w_j}) \geq g_w > g_k \geq \min\{f(X) : X \in T^k\}$$

$$= f(\hat{X}). \quad (\text{by the hypothesis})$$

That is,

$$f(X_{w_j}) > f(\hat{X}), \quad X_{w_j} \in X_w, \quad w \geq k + 1. \quad (3)$$

Eqs. (2) and (3) imply that  $f(\hat{X})$  is the least among the values of  $f(X)$  at all the integer feasible solutions in  $S$ . Thus  $\hat{X}$  is an optimal solution of QIP and  $f(\hat{X}) = f_1$ .

Notice that

$$L_1 = \{X \in T^k : f(X) = f(\hat{X})\}$$

is the set of the optimal feasible solutions of QIP.

**Corollary 2.2.** *If*

$$g_1 = \min\{f(X) : X \in T^1\} = f(\hat{X}),$$

*then  $\hat{X}$  is an optimal solution of QIP and*

$$\{X \in T^1 : f(X) = g_1\}$$

*is the set of the optimal solutions of QIP.*

The following Remark 2.3 is on the current lower and upper bounds on the optimal objective function value of QIP.

**Remark 2.3.** *If*

$$g_k < \min\{f(X) : X \in T^k\},$$

*then*

$$g_k < f_1 \leq \min\{f(X) : X \in T^{k+1}\}.$$

Proposition 2.4 established below pertains to the  $k$ -th ( $k \geq 2$ ) best integer feasible solution of QIP.

**Proposition 2.4.** *If*

$$g_p \geq \min\{f(X) : f(X) > f_{k-1}, X \in T^p\} = f(X^*),$$

*say, then  $X^*$  is one of the  $k$ -th best integer feasible solutions of QIP.*

**Proof.**

$$f(X^*) = \min\{f(X) : f(X) > f_{k-1}, X \in T^p\}.$$

Therefore, it follows that for those  $X$  in  $T^p$  for which  $f(X) > f_{k-1}$ , we have

$$f(X) \geq f(X^*). \quad (4)$$

Also,

$$f(X_{q_j}) \geq g(X_{q_j}) = g_q, \quad X_{q_j} \in X_q. \quad (\text{by (1)})$$

Therefore,

$$\begin{aligned} f(X_{q_j}) &\geq g_q > g_p, \quad q \geq p + 1 \\ &\geq f(X^*). \quad (\text{by the hypothesis}) \end{aligned}$$

That is,

$$f(X_{q_j}) > f(X^*), \quad X_{q_j} \in X_q, \quad q \geq p + 1. \quad (5)$$

Eqs. (4) and (5) imply that  $X^*$  is a  $k$ -th best integer feasible solution of QIP.

Notice that

$$L_k = \{X \in T^p : f(X) = f(X^*)\}$$

is the set of all the  $k$ -th best integer feasible solutions of QIP.

Similar to Remark 2.3 the following Remark 2.5 is on the current bounds on the  $k$ -th best value of the objective function in QIP.

**Remark 2.5.** *If*

$$g_p < \min\{f(X) : f(X) > f_{k-1}, X \in T^p\},$$

*then*

$$g_p < f_k \leq \min\{f(X) : f(X) > f_{k-1}, X \in T^{p+1}\}.$$

**Remark 2.6.** Suppose the set of the last best integer feasible solutions of LIP is reached and  $f_1, f_2, \dots, f_i$  have been computed. Then the next best values of  $f(X)$  are given by

$$f_{i+a} = \min\{f(X) : f(X) > f_{i+a-1}, X \in T^N\},$$

$$a \geq 1,$$

and

$$L_{i+a} = \{X \in T^N : f(X) = f_{i+a}\}, \quad a \geq 1.$$

### 3. Algorithm

*Algorithm for ranking in the Quadratic Integer Programming (QIP) problem*

The ranking algorithm comprises of the following steps:

*Initial step.* Find  $U_j$  by solving

$$\text{Min}_{X \in S} X^T D_j, \quad j = 1, 2, \dots, n,$$

and construct the Linear Integer Programming (LIP) problem.

*Step 1.* (Search for the best solutions of QIP.)

*Step 1(a).* Solve LIP and find  $X_1 = T^1$ , the set of its optimal integer feasible solutions (Verma et al. [19]; Salkin [18]).

Compute  $g_1$  and  $f(X)$ ,  $X \in T^1$ .

If

$$g_1 = \min\{f(X) : X \in T^1\} = f(\hat{X}),$$

say, then  $\hat{X}$  is an optimal solution of QIP and the corresponding optimal value is  $f(\hat{X})$ .

$$L_1 = \{X \in T^1 : f(X) = f(\hat{X})\}.$$

If

$$g_1 < \min\{f(X) : X \in T^1\},$$

then set  $s = 2$  and go to Step 1(b).

*Step 1(b).* Find  $X_s$  and  $g_s$  ( $s \geq 2$ ) (Verma et al. [19]).

If

$$g_s \geq \min\{f(X) : X \in T^s\} = f(\hat{X}),$$

say, then  $\hat{X}$  is an optimal solution of QIP and the corresponding optimal value is  $f(\hat{X})$  (by Proposition 2.1).

$$L_1 = \{X \in T^s : f(X) = f(\hat{X})\}.$$

If

$$g_s < \min\{f(X) : X \in T^s\},$$

repeat Step 1(b) for the next higher value of  $s$ .

*General step.* (Search for the  $k$ -th best solutions of QIP,  $k \geq 2$ .) Find  $X_s$  and  $g_s$  ( $s \geq 2$ ).

If

$$g_s \geq \min\{f(X) : f(X) > f_{k-1}, X \in T^s\} = f(X^*),$$

say, then  $X^*$  is a  $k$ -th best integer feasible solution of QIP and  $f(X^*)$  is the  $k$ -th best objective function value (by Proposition 2.4).

$$L_k = \{X \in T^s : f(X) = f(X^*)\}.$$

If

$$g_s < \min\{f(X) : f(X) > f_{k-1}, X \in T^s\},$$

repeat this step for the next higher value of  $s$ .

*Terminal step.* Suppose  $g_N$  and  $X_N$  are reached and  $f_1, f_2, \dots, f_t$  have already been computed. Then

the next best values  $f_{t+a}$  ( $a \geq 1$ ) of  $f(X)$  in QIP are given by

$$f_{t+a} = \min\{f(X) : f(X) > f_{t+a-1}; X \in T^N\},$$

$$a \geq 1, \quad (\text{by Remark 2.6})$$

and

$$L_{t+a} = \{X \in T^N : f(X) = f_{t+a}\}, \quad a \geq 1.$$

*Concluding remarks*

(i) To obtain the best (optimal) integer feasible solution of problem (P), ranking the integer feasible solutions of the QIP problem in non-decreasing order of the values of  $f(X)$  is carried up to a stage where an integer feasible solution of QIP is obtained which satisfies the additional complicated nonlinear constraints  $h(X) \leq 0$ .

(ii) It may be observed that the integer feasible solutions of the QIP problem are ranked by ranking the integer feasible solutions of the associated LIP problem. For ranking the integer feasible solutions in non-decreasing order of the values of the objective function in LIP, one is referred to the ranking approach developed by Verma et al. [19]. In Verma's approach edge truncating cuts are introduced successively to discard the current integer feasible solutions (if found not suitable by the decision maker) of LIP and then standard methods (like Gomory's cutting plane technique) are used to find the next best integer feasible solution of LIP. The sensitivity analysis approach of appending a constraint is used to obtain the new integer feasible solution and thus one is not required to solve a new LIP afresh. It is only the original LIP which is constantly under study and post-optimality analysis helps in finding the next best integer feasible solution of LIP. Otherwise also, we are not aware of any other method of ranking the integer feasible solutions of LIP except that developed by Verma et al. [19].

As mentioned earlier, ranking the integer feasible solutions of a nonlinear programming problem and in particular QIP has not yet been taken up. This is the main motivation for the development of the algorithm stated above.

(iii) Problem (P) studied in the present paper can, perhaps, be solved only by the proposed algorithm of ranking the integer feasible solutions of QIP. Finding

out more efficient solution methodologies for such problems may be a motivating force for researchers to take up this study.

(iv) The solution methodologies for the QIP problem mentioned earlier in Section 1 do not involve the ranking of its integer feasible solutions. These procedures obtain only its optimal integer feasible solution. For the time being, we are also unable to suggest any other methodology for solving problem (P) wherein the complicated nonlinear constraints are present. The main aim of the present study is to throw open in this way the very realistic problem and suggest an algorithm for its solution. It is hoped that these ideas will stimulate more research in this direction.

Also, as there is no other solution methodology available, we are unable to present any comparison. (v) Note that the form of the matrix  $D$  is not crucial to the method. Any one of several equivalent ways of writing a quadratic form could be used. The form chosen would depend on the ease of computing the  $U_j$ 's and the formulation of LIP.  $D$  may not even be a positive/negative (semi-)definite matrix.

#### 4. Numerical illustration

**Example.** Consider the problem

(P)

$$\begin{aligned} \text{Min } f(X) &= 5x_1 + 12x_2 - 2x_1^2 - x_2^2 \\ \text{subject to } 2x_1 + x_2 &\leq 10, \\ 4x_1 + 5x_2 &\geq 20, \\ x_1, x_2 &\geq 0 \text{ and integers,} \end{aligned}$$

where  $X = (x_1, x_2)^T$  satisfies the additional constraint

$$2x_1 + 8x_2 - 2x_2^2 \geq 15.$$

Here QIP is

$$\text{Min}_{X \in S} 5x_1 + 12x_2 - 2x_1^2 - x_2^2,$$

where

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 + x_2 \leq 10, 4x_1 + 5x_2 \geq 20; x_1, x_2 \geq 0 \text{ and integers}\}$$

Note that

$$D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}.$$

To rank the integer feasible solutions of QIP, we find  $(U_1, U_2)$  and construct the LIP problem.

$$U_1 = \min_{X \in S} (-2x_1) = -10,$$

$$U_2 = \min_{X \in S} (-x_2) = -10.$$

Thus the related linear integer programming (LIP) problem is

(LIP)

$$\text{Min}_{X \in S} g(X) = -5x_1 + 2x_2,$$

where  $X_1 (= T^1)$  = (the set of the optimal integer feasible solutions of LIP) =  $\{X_{1_1} = (5, 0)\}$ ,  $g_1 = -25$ , and  $f(X_{1_1}) = -25$ .

Thus

$$g_1 = \min\{f(X) : X \in T^1\} = f(X_{1_1}) = -25.$$

Therefore, the optimal (best) integer feasible solution of QIP is  $X_{1_1} = (5, 0)$ . As it does not satisfy

$$2x_1 + 8x_2 - 2x_2^2 \geq 15,$$

proceed to find the second best integer feasible solutions of QIP.

For the second best integer feasible solutions of QIP, find the set  $X_2$  of the second best integer feasible solutions of LIP:

$$X_2 = \{X_{2_1} = (4, 1)\}, \quad g_2 = -18,$$

as

$$\begin{aligned} g_2 &< \min\{f(X) : f(X) > f_1, X \in T^2\} \\ &= f(X_{2_1}) = -1. \end{aligned}$$

Proceed to find the next best integer feasible solutions of LIP using the procedure developed by Verma et al. [19].

$$X_3 = \{X_{3_1} = (4, 2)\}, \quad g_3 = -16,$$

$$f(X_{3_1}) = 8, \quad T^3 = \bigcup_{i=1}^3 X_i.$$

$$X_4 = \{X_{4_1} = (3, 2)\}, \quad g_4 = -11,$$

$$f(X_{4_1}) = 17, \quad T^4 = \bigcup_{i=1}^4 X_i.$$

$$X_5 = \{X_{5_1} = (3, 3)\}, \quad g_5 = -9,$$

$$f(X_{5_1}) = 24, \quad T^5 = \bigcup_{i=1}^5 X_i.$$

$$X_6 = \{X_{6_i} = (3, 4)\}, g_6 = -7, \\ f(X_{6_i}) = 29, T^6 = \bigcup_{i=1}^6 X_i.$$

$$X_7 = \{X_{7_i} = (2, 3)\}, g_7 = -4, \\ f(X_{7_i}) = 29, T^7 = \bigcup_{i=1}^7 X_i.$$

$$X_8 = \{X_{8_i} = (2, 4)\}, g_8 = -2, \\ f(X_{8_i}) = 34, T^8 = \bigcup_{i=1}^8 X_i.$$

$$g_8 < \min\{f(X) : f(X) > f_1, X \in T^8\}.$$

$$X_9 = \{X_{9_i} = (2, 5)\}, g_9 = 0, \\ f(X_{9_i}) = 37, T^9 = \bigcup_{i=1}^9 X_i.$$

$$g_9 > \min\{f(X) : f(X) > f_1, X \in T^9\} = f(X_{2_1}).$$

Therefore, the second best integer feasible solution of QIP is  $X_{2_1} = (4, 1)$ . Again as  $(4, 1)$  does not satisfy the constraint

$$2x_1 + 8x_2 - 2x_2^2 \geq 15,$$

proceed to find the third best integer feasible solution of QIP. Proceeding as explained above, the third best integer feasible solution of QIP is  $X_{3_1} = (4, 2)$ . As this third best integer feasible solution of QIP satisfies  $2x_1 + 8x_2 - 2x_2^2 \geq 15$ , the optimal integer feasible solution of problem (P) is  $(4, 2)$  and the optimal value is 8.

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