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EXTREME POINT ENUMERATION TECHNIQUE FOR ASSIGNMENT PROBLEM

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ABSTRACT. In this paper an extreme point enumeration technique is developed for solving assignment problem. The procedure starts from an extreme point (infeasible) of a unit cube and moves from one extreme point to another extreme point in such a way that feasibility and optimality for the problem are achieved simultaneously. A numerical eaxmple is provided in support of the theory.

Theoretical Development

Consider an assignment problem in which there are n facilities (say persons) and n jobs. One job is to be assigned to one facility only. No job is to be left undone and each facility is to be completely used. The mathematical model (see reference [11]) for such a problem is:

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}\, x_{ij}$$

$$\left.\begin{aligned} \text{such that } \sum_{i} x_{ij} &= 1 \\ \sum_{j} x_{ij} &= 1 \end{aligned}\right| \quad \ldots \quad \ldots \quad \ldots \qquad (1.1)$$

$$x_{ij} = 0 \text{ or } 1 \text{ according as } i^{th} \text{ facility doesn't or does work on } j^{th} \text{ job} \quad \Big| \quad \ldots \qquad (1.2)$$

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where C_{ij} is the effectiveness of i^{th} facility on j^{th} job (for example C_{ij} may be the cost involved in assigning j^{th} job to i^{th} person), x_{ij} is the fraction of time for which i^{th} facility works on j^{th} job. (Here we assume optimisation to be minimisation, for example, we want to minimise the total cost of assignment).

The assignment problem is clearly a particular type of zero-one integer programming problem but its structure is of very special type. This special nature of the problem is given by the equation (1.1) which in turn mean that

$$\left.\begin{array}{l} \text{if } x_{kl} = 1,\ x_{il} = 0,\ i = 1, 2, \ldots n;\ i \neq k \\ \text{and if } x_{pq} = 1,\ x_{pj} = 0,\ j = 1, 2, \ldots n,\ j \neq q \end{array}\right| \quad \ldots \quad (1.3)$$

Also as one facility is to work on one job only, it follows that there should be only $n - x_{ij}$'s which are unity each and remaining zeros and (1.3) shows that in this set of $n - x_{ij}$'s no two i's should be same and no two j's should be same. This will be treated as feasibility criterion for the problem.

Fron zero-one constraint in (1.2) it is clear that our solution is going to be an extreme point of a unit cube in n^2-dimensions. As we are interested in only n-x_{ij}'s which are unity each, we will consider only those extreme points of the cube which have n out of n^2-components as unity and remaining zeros. From one such extreme point we move to another extreme point giving least possible rise in the value of the objective function (i. e., we move to the next best extreme point). At each step solution obtained is tested for feasibility and the moment it is achieved we stop and determine the optimal allocation.

Let us explain as to how we determine the initial basic solution (which is to be be an extreme point of a unit cube having only n of its components as unity). Determine Z_l defined as:

$$\text{Max}\left[\sum_i \underset{j}{\text{Min}}\ (C_{ij}),\ \sum_j \underset{i}{\text{Min}}\ (C_{ij})\right] = Z_l \text{ , say}$$

$$\text{If } Z_l = \sum_{i=1}^{n} \underset{j=1,2,\ldots n}{\text{Min}}\ (C_{ij})$$

$$= \underset{j}{\text{Min}}\ (C_{1j}) + \underset{j}{\text{Min}}\ ({}_{2j}) + \ldots + \underset{j}{\text{Min}}\ (C_{nj})$$

$$= \sum_{i=1}^{n} C_{im_i} \text{ (say)}$$

where $C_{im_i} = \min_j (C_{ij})$ and $m_i \in [1, 2, \ldots, n]$, then the corresponding extreme point of the unit cube which provides an initial basic solution is given as:

$$x_{1m_1} = 1, x_{2m_2} = 1, \ldots\ldots x_{nm_n} = 1$$

remaining x_{ij}'s zeros.

Similarly if $Z_l = \sum_j \min_i (C_{ij})$, the corresponding extreme point can be obtained. Clearly Min $Z \geqslant Z_l$. This extreme point is tested for feasibility and if it is satisfied we get optimal solution otherwise we proceed further.

We observe that if $Z_l = \sum_i \min_j (C_{ij})$ and the corresponding extreme point solution satisfies feasibility then if the extreme point solution corresponding to $\sum_j \min_i (C_{ij})$ is also feasible then $\sum_j \min_i (C_{ij}) = = \sum_i \min_j (C_{ij}) = Z_l$ and if extreme point solution corresponding to $\sum_j \min_i (C_{ij})$ is infeasible then $\sum_j \min_i (C_{ij}) < \sum_i \min_j (C_{ij})$.

For the purpose of a check we can also determine a rough upper bound beyond which Z should not go. Determine any solution feasible in the sense explained and let corresponding value of Z be Z_u. Clearly Min $Z \leqslant Z_u$. For example the simplest way of getting Z_u is given by the following equation:

$$Z_u = \min \left[\sum_{i=1}^{n} C_{ii}, \sum_{i=1}^{n} C_{i,n+1-i} \right]$$

The procedure will definitely converge in finite number of steps because of the following facts:

i) procedure moves from extreme point to extreme point of the unit cube and the number of these extreme points is firite viz. 2^{n^2}.

ii) only those extreme points are considered which have n out of n^2-components as unity and remaining zeros.

iii) no extreme point is repeated as at each step value of the objective function is to be improved.

In solving problems we arrange C_{ij}'s in ascending order (if optimisation is minimisation). Z_l is determined and the corresponding extreme point of the unit cube is found. From this extreme point we move to another extreme point (having only n of its components as unity) giving least rise in Z till feasibility is achieved. Care is taken that we never cross Z_u.

REMARKS:

1. As we have to deal with different arrangements of n unities, it is expected that the present approach will be more efficient on computers.
2. In 'Zero-One Integer Programming' we generally move from fesible solution and continue till optimality is reached. Here we move from an infeasible solution in such a way that we reach feasibility and optimality simultaneously.
3. In this paper the main emphasis has been on a different approach rather than on analysing the computational efficiency of it.

EXAMPLE. The cost associated with allocating each of the four facilities F_1, F_2, F_3 and F_4 to each of the four jobs J_1, J_2, J_3 and J_4 are given below in the effectiveness matrix $\|C_{ij}\|$. Assign each facility to a job in such a way that the total cost of assignment is minimised.

Jobs / Facility	J_1	J_2	J_3	J_4
F_1	1	8	4	1
F_2	5	7	6	5
F_3	3	5	4	2
F_4	3	1	6	3

Mathematical model for the problem is:

to find $x_{ij} \geqslant 0,\ i, j = 1, 2, \ldots 4$

$$\text{such that } \sum_i x_{ij} = 1$$

$$\sum_j x_{ij} = 1$$

$$x_{ij} = 0 \text{ if } i^{th} \text{ facility works on } j^{th} \text{ job}$$

$$= 1 \text{ if } i^{th} \text{ facility does not work on } j^{th} \text{ job}$$

$$\text{and Min} \sum_i \sum_j C_{ij}\, x_{ij}$$

where $\|C_{ij}\|$ is the given 4×4 matrix.

$$Z_l = \text{Max}\left[\sum_i \underset{j}{\text{Min}}\,(C_{ij}),\ \sum_j \underset{i}{\text{Min}}\,(C_{ij})\right]$$

$$= \text{Max}\,[(1 + 5 + 2 + 1), (1 + 1 + 4 + 1)]$$

$$= \text{Max}\ [9, 7] = 9.$$

9 is yielded by $x_{11} = 1$, $x_{21} = 1$, $x_{34} = 1$, $x_{42} = 1$, remaining x_{ij}'s zeros.

$$Z_u = \text{Min}\,[(1 + 7 + 4 + 3), (1 + 6 + 5 + 3)] = 15$$

Therefore, $9 \leqslant \text{Min Z} \leqslant 15$.

Our starting value will be $Z_l = 9$ and starting extreme point of unit cube will be $x_{11} = 1$, $x_{21} = 1$, $x_{34} = 1$, $x_{42} = 1$ remaining x_{ij}'s zeros.

As feasibility is not achieved here we move to the next extreme point giving least rise in Z. For this C_{ij}'s are arranged in ascending order. The whole movement from extreme point to extreme point of the unit cube is explained in the table.

C_{ij}	x_{ij}	Extreme Points of Unit Cube									
1	x_{11}	1	1	1	1	0	1	0	0	0	1
1	x_{14}	0	0	0	0	1	0	1	0	1	0
1	x_{42}	1	1	1	1	1	0	0	1	1	1
2	x_{34}	1	1	1	0	0	1	1	1	1	1
3	x_{31}	0	0	0	0	1	1	1	1	0	0
3	x_{41}	0	0	0	1	0	0	0	0	0	0
3	x_{44}	0	0	0	0	0	1	1	1	0	0
4	x_{13}	0	0	1	0	0	0	0	0	0	0
4	x_{33}	0	0	0	0	1	0	0	0	0	0
5	x_{21}	1	0	0	0	0	0	0	0	0	0
5	x_{24}	0	0	1	0	0	0	0	0	0	0
5	x_{32}	0	1	0	0	0	0	0	0	0	0
6	x_{23}	0	0	0	0	0	0	0	0	1	1
6	x_{43}	0	0	0	0	0	0	0	0	0	0
7	x_{22}	0	0	0	0	0	0	0	0	0	0
8	x_{12}	0	0	0	0	0	0	0	0	0	0
	Z →	9	9	9	9	9	9	9	9	10	10
	Feasible (√) Infeasible (X)	X	X	X	X	X	X	X	X	X	√

Thus optimal solution is: $x_{11} = 1$, $x_{23} = 1$, $x_{34} = 1$, $x_{42} = 1$ remaining x_{ij}'s zeros.

and Min Z = 10

Therefore, optimal allocation is:

1st facility should work on 1st job
2nd facility should work on 3rd job
3rd facility should work on 4th job
4th facility should work on 2nd job

NOTE:

1. Some of the alternative extreme points yielding value 9 have been omitted in the table because their non-feasibility is quite apparent. For example $6 + 1 + 1 + 1 = 9$ yielded by $x_{23} = 1$, $x_{11} = 1$,

$x_{14} = 1$, $x_{42} = 1$ and remaining x_{ij}'s zeros is clearly non-feasible. Observe that from each set of bracketed variables we can have only one variable at a time. Variables taken from the same bracket are, apparently, non-feasible.

2. Sometimes the problem is simplified by using the following theorem: 'If in an assignment problem we add a constant to every element of a row (or column) in the effectiveness matrix, then an assignment that minimizes the total effectiveness in one matrix also minimizes the total effectiveness in the other matrix'. (See reference [2]).

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