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Maximizing pseudoconvex transportation problem: a special type

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Abstract. The paper discusses a non-concave fractional programming problem aiming at maximization of a pseudoconvex function under standard transportation conditions. The pseudoconvex function considered here is the product of two linear functions contrasted with a positive valued linear function. It has been established that optimal solution of the problem is attainable at an extreme point of the convex feasible region. The problem is shown to be related to 'indefinite' quadratic programming which deals with maximization of a convex function over the given feasible region. It has been further established that the local maximum point of this quadratic programming problem is the global maximum point under certain conditions, and its optimal solution provides an upper bound on the optimal value of the main problem. The extreme point solutions of the 'indefinite' quadratic program are ranked to tighten the bounds on the optimal value of the main problem and a convergent algorithm is developed to obtain the optimal solution.

Zusammenfassung. In der Arbeit betrachten wir ein Transportproblem mit nicht-konkaver, peudo-konvexer Zielfunktion, die sich als Quotient des Produktes zweier linearer Funktionen und einer linearen Funktion ergibt. Man kann zeigen, daß die Optimallösung für dieses Problem in einer Ecke des zulässigen Bereichs angenommen wird. Die betrachtete Problemstellung ist verwandt mit der Problemstellung der indefiniten quadratischen Optimierung. Für diese Probleme ist jedes lokale Optimum auch global optimal und die Optimallösung liefert zunächst eine obere Schranke für unser Ausgangsproblem. Durch ein „Ranking" der Ecken des Quadratischen Programms erhalten wir dann die Optimallösung für das pseudo-konvexe Transportproblem.

Key words: Non-concave fractional programming, transportation problem, ranking in 'indefinite' quadratic programming

Schlüsselwörter: Pseudo-konvexe Optimierung, Transportproblem, Enumeration

1. Introduction

The objective to attain optimum level of efficiency in every sphere of activity has given rise to fractional programming. Examples are many: optimizing productivity of material in industry plant [3], maximizing ratio of return and cost in resource allocation problems [11], of profit per unit of time in cargo loading problem [10], of expected return and risk in portfolio selection problem [20], etc. One can find a galaxy of such problems in the review prepared by Schiable [15].

Depending on the nature of the functions whose ratios are to be optimized, different types of fractional programs arise. As for example where the ratio of linear functions is optimized one requires to develop a linear fractional programs. On the other hand he would deal with a quadratic or convex-concave fractional programs when the ratio of quadratic functions or convex-concave functions is optimized. Programs of the latter type are termed in the literature as non-concave fractional programs.

It is observed that most of the algorithms known so far solve linear or more generally concave fractional programs as discussed by Schiable [16]. And to a much lesser degree solution methods are available for non-concave fractional programs where the ratio of two concave, two convex or the ratio of a convex and a concave function is to be maximized [16]. This paper develops an algorithm for non-concave fractional program of convex-concave type.

Section 2 gives the mathematical model of the problem and its practical application, while Sect. 3 discusses the limitations of the available methods and the approach followed. The last section develops the theory and the algorithm.

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2. Problem and practical application

Mathematical model of the problem considered here is:

$$\operatorname*{Max}_{X \in S} Z(X) = \frac{(C^T X + \alpha)(D^T X + \beta)}{(E^T X + e)} \qquad \text{(P1)}$$

where,

$$S = \left\{x_{ij} \in R^{mn}: \sum_J x_{ij} = a_i;\ i \in I;\ \sum_I x_{ij} = b_j,\ j \in J;\ x_{ij} \geq 0\right\}$$

$X = (x_{ij})$; $C = \{c_{ij}\}$; $D = \{d_{ij}\}$; $E = \{e_{ij}\}$ are $mn \times 1$ vectors $(C^T X + \alpha) \geq 0$, $(D^T X + \beta) > 0$, $(E^T X + e) > 0$ for all $X \in S$ (1)

$(C^T X + \alpha)$ and $(D^T X + \beta)$ are of non-conflicting nature (2)

This model is expected to have potential applications in many practical situations. Selection of retailers is one such situation which is described below:

Consider the set I of government agencies and a set J of wholesalers who receive goods from these agencies and in turn supply goods to the retailers R_{ij}.

Let d_{ij} be the per unit return of the jth wholesaler on goods received from the ith agency,
- e_{ij} be the per unit transportation cost (or depreciation) of the goods received by the jth wholesaler from the ith agency, which in turn are supplied to retailer R_{ij},
- c_{ij} be the per unit profit of the retailer R_{ij} per unit earnings of the wholesalers,
- x_{ij} be the quantity received by the jth wholesaler from the ith government agency which is in turn supplied to retailer R_{ij}.

Clearly, $\left(\sum_I \sum_J c_{ij} x_{ij}\right)$ and $\left(\sum_I \sum_J d_{ij} x_{ij}\right)$ will increase or decrease together.

Aim is to decide which retailers R_{ij} are advisible to be allowed to function so as to maximize the total profit of the retailers per unit of net transportation cost (or depreciation).

That is,

$$\operatorname*{Max}_{X \in S} \frac{\sum_I \sum_J \left[c_{ij} x_{ij} \left(\sum_I \sum_J d_{ij} x_{ij}\right)\right]}{\left(\sum_I \sum_J e_{ij} x_{ij}\right)}$$

The solution will tell as to which retailers are desirable to be kept functioning. In the current case, one of the best set of $(m+n-1)$ retailers would be found whose functioning will maximize the aforesaid objective function.

3. Available methods, limitations and our approach

The problem (P1) being a non-linear programming problem can be solved by methods of feasible directions like Zoutendijk's method, Rosen's gradient method, Wolfe's method of reduced gradient, Zangwill's convex-simplex method etc. [19]. In these feasible direction algorithms, after knowing a feasible point X a direction d is determined and a one-dimensional optimization problem is solved to find the step length $\lambda (>0)$ such that $\hat{X} = X + \lambda d$ is a better feasible point. The process is repeated as often as possible. But since optimal solution to (P1) as shown in section 4, lies at an extreme point of S, one can avoid the search for a feasible direction and do away with solving a one-dimensional optimization problem to find step length $\lambda (>0)$ in the chosen direction. In problems of the type (P1) one is motivated to move from one extreme point to a better extreme point and continue the systematic ranking of the extreme points of S till optimal solution is obtained.

One could also think of applying Gould's [1, 2] methodology involving simple differentiable quadratic penalty functions $p(X, v)$ defined as:

$$p(X, v) = f(X) + \frac{1}{2v} \sum_{I_1} g_i(X)^2 + \frac{1}{2v} \sum_{I_2} [\min(0, g_i(X))]^2$$

in minimization of smooth non-linear functions $\operatorname*{Min}_X f(X)$ subject to general constraints: $g_i(X) = 0$, $i \in I_1$ and $g_i(X) \geq 0$, $i \in I_2$.

These methods though known for many years, have however been disregarded on account of natural ill-conditioning of the penalty function when the penalty parameter shrinks to zero. Since linear constraints can be efficiently handled by various other methods, their incorporation in the penalty function is not particularly useful.

Standard 'barrier function methods' are also not advisable, as the constraints in (P1) are equality constraints.

Higgins and Polak's [4] approach, which is an extension of the Von Hohenbalken's algorithm [18] obtains the global minimum solution for a pseudoconvex function. This solution strategy can not be applied to solve the problem (P1) as it deals with maximization of a pseudoconvex function over a convex compact polytope where local maximum point may not be global. As the known methods appear to be of little use for obtaining global maximum solution of a non-concave fractional program exploiting transportation structure of the constraint set this paper develops a novel approach for its solution based upon systematic ranking of extreme point solutions.

4. Theoretical development

Definition. Non-conflicting functions. *Functions f and g are said to be of non-conflicting nature, if*

$$f(X^1) > f(X^2) \leftrightarrow g(X^1) > g(X^2).$$

It can be proved that for affine functions $(C^T X + \alpha)$ and $(D^T X + \beta)$ this condition holds iff $D^T = \lambda C^T$, where $\lambda > 0$.

The problem (P1) therefore, gets restructured to (P1)′:

$$\text{(P1)}': \operatorname*{Max}_{X \in S} Z(X) = \frac{(C^T X + \alpha)(\lambda C^T X + \beta)}{(E^T X + e)}$$

(P1)′ is closely related to the problem (P2) defined as:

$$\text{(P2)}: \operatorname*{Max}_{X \in S} U(X) = (C^T X) + \alpha)(\lambda C^T X + \beta).$$

Some remarks

(i). $U(X)$ is a convex function and therefore the maximum value in (P2) is attainable at an extreme point of S [13, 19]. This motivates the investigation of extreme point solutions of S while finding the optimal solutions of (P2). The local maximum point reached by the solution methodology for (P2) will be the global maximum point as $U(X)$ is pseudoconcave as well under the assumption (1) specified in section 2 [13, 19].

(ii). $Z(X) = \dfrac{(C^T X + \alpha)(\lambda C^T X + \beta)}{(E^T X + e)}$ under the assumption (1) is pseudoconvex [13, 19]. This pseudoconvex function $Z(X)$ defined on a polyhedral region S attains its maximum on a vertex, since it is known that if $\triangleleft Z(X) = 0$ at an interior point, then X is a minimum point. So unless Z is constant, a maximum must happen on the boundary. Applying the same idea to faces and edges, the maximum is located at a vertex of the polyhedron S. This motivates the systematic scanning of extreme points of S for obtaining an optimal solution of (P1)′.

(iii). $Z(X) < \dfrac{U(X)}{r}$ for all $X \in S$.

where $r = \operatorname{Min}_{X \in S} (E^T X + e)$.

Notations

$\bar{S}$: the set of extreme points of S,
S_i: the set of ith best extreme point solutions of (P2),
$U_i \equiv U(X)$, $X \in S_i$.

Definition. k^{th} best extreme point solution of (P2). *X^k is a k^{th} best extreme point solution of* (P2) *if*

$$U(X^k) \equiv U_k = \operatorname{Max}\left[U(X): X \in \bar{S} \setminus \bigcup_{i=1}^{k-1} S_i\right].$$

Theorem 1. If $\dfrac{U_{k+1}}{r} \leq \operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k} S_j\right] < \dfrac{U_k}{r}$, *then the optimal value of* $Z(X)$ *in* (P1)′ *is*

$$\operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k} S_i\right].$$

Proof. For $s > k+1$,

$$\frac{U_s}{r} < \frac{U_{k+1}}{r}$$
$$\leq \operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k} S_i\right] \text{ (by the hypothesis).}$$

Also for all $X \in S_s$, $Z(X) \leq \dfrac{U_s}{r}$

$$< \operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k} S_i\right].$$

Therefore, no extreme point of S after the $(k+1)^{\text{th}}$ best extreme point solution of (P2) can yield objective function value of (P1)′ better than $\operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k} S_i\right]$. Thus, $\operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k} S_i\right]$ is the optimal value of the objective function in (P1)′.

If $\operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k} S_i\right] = Z(X^{\nu})$, then X^{ν} is the optimal solution of (P1)′.

Algorithm for finding global optimal solution of problem (P1)′

Global optimal solution of (P1)′ can be obtained by successively tightening the bounds on its objective function value. This is achieved by ranking the extreme point solutions of problem (P2) is descending order of $U(X)$ values [see appendix].

The steps involved in the algorithm are described as under:

Step 1: Solve $\operatorname{Min}_{X \in S} (E^T X + e)$ to find its minimum value r.

Step 2: Find S_1, the set of optimal solutions of (P2) [5–8]. If, $\dfrac{U_1}{r} = \operatorname{Max}[Z(X): X \in S_1] \equiv Z(X^1)$ (say), stop. $X^1 \in S_1$ is an optimal solution of (P1)′. Otherwise go to step 3.

Step 3: $(k \geq 2)$. Find the set S_k of the k^{th} best extreme point solutions of (P2) [see appendix].

If either (a) $S_k \neq \emptyset$ and $\dfrac{U_k}{r} \leq \operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k-1} S_i\right] \equiv Z(X_{\nu})$ or (b) if $S_k = \emptyset$, $U_{i+1} < U_i$, $i = 1, 2, \ldots, k-2$, and $\dfrac{U_{k-1}}{r} > \operatorname{Max}\left[Z(X): X \in \bigcup_{i=1}^{k-1} S_i\right] \equiv Z(X_{\eta})$, then go to step 4.

Otherwise, replace k by $k+1$ and return to step 3.

Step 4: Algorithm ends yielding an optimal solution X^{ν} if 3(a) holds, or X^{η} if 3(b) holds.

Convergence of the algorithm

Algorithm obtains the optimal solution of (P1)′ in a finite number of iterations as only the extreme point solutions of (P2) are investigated systematically.

Appendix: Ranking of extreme point solutions of (P2)

Step 1: To find optimal basic feasible solutions (OBFS's) of (P2)

The set S_1 of the optimal basic feasible solutions of (P2) can be obtained by one of the four methods developed by Swarup [5–8].

If B_1^l is the set of basic cells in a basic feasible solution of (P2), then this will be an optimal solution if

$$R_{1,ij}^l \leq 0 \quad \forall \text{ cells } (i,j) \notin B_1^l$$
$$= 0 \quad \forall \text{ cells } (i,j) \in B_1^l$$

where

$R^l_{1,ij} = e_{ij}\lambda(Z_{ij}-c_{ij})^2 - \lambda f^l_1(Z_{ij}-c_{ij}) - g^l_1(Z_{ij}-c_{ij})$

$u_i + v_j = c_{ij} \quad \forall$ cells $(i,j) \in B^l_1$

$u_i + v_j = Z_{ij} \quad \forall$ cells $(i,j) \notin B^l_1$

and,

f^l_1 = value of $(C^T X + \alpha)$ at the current basic feasible solution corresponding to B^l_1.

g^l_1 = value of $(\lambda C^T X + \beta)$ at the current basic feasible solution corresponding to B^l_1.

θ_{ij} is the level at which the non-basic cell (i,j) enters the basis B^l_1, replacing some basic cell $\in B^l_1$.

Note that u_i, v_j, θ_{ij} are determined by using the standard Stepping-Stone algorithm for balanced transportation problem.

Step 2: To find the 2nd best basic feasible solution of (P2)

Basic concepts of finding the 2nd best basic feasible solution of (P2) are similar to those given in [9, 12, 14, 17].

Construct the set H_1 defined as follows:

$$H_1 = \bigcup_l \{(i,j): R^l_{1,ij} < 0, (i,j) \notin B^l_1\}.$$

Find $\max_{\langle i,j\rangle \in H_1} [\theta_{ij} R^l_{1,ij}] \equiv \theta_{st} R^h_{1,st}$ (say).

Then entry of non-basic cell $(s,t) \notin B^h_1$ at level θ_{st}, replacing some basic cell in B^h_1 yield 2nd best basic feasible solution of (P2).

The corresponding 2nd best value of $U(X)$ is $U_2 = (U_1 + \theta_{st} R^h_{1,st})$.

If $\max_{\langle i,j\rangle \in H_1} [\theta_{ij} R^l_{1,ij}]$ is obtained for only one cell $(i,j) \in H_1$, then 2nd best basic feasible solution is unique, otherwise there may be more than one 2nd best basic feasible solution.

Step 3: To find the $(k+1)^{\text{th}}$ best basic feasible solutions of (P2), $(k \geq 2)$ [9, 12, 14, 17]

Supposing that the basic feasible solutions of (P2) up to the k^{th} best solution have been obtained, and B^l_k be the set of basic cells in a k^{th} best basic feasible solution of (P2).

Construct the set H_k defined as:

$$H_k = \bigcup_l \{i,j): R^l_{k,ij} < 0, (i,j) \notin B^l_k, B^l_k \text{ is the set of basic cells in a } k^{\text{th}} \text{ best basic feasible solution of (P2)}\}.$$

Find

$$\max\left[\max_{\langle i,j\rangle \in \bar{H}_1} (U_1 + \theta_{ij} R^l_{1,ij}), \ldots, \max_{\langle i,j\rangle \in \bar{H}_k} (U_k + \theta_{ij} R^l_{k,ij})\right]$$

where

$$\bar{H}_q = H_q \setminus \left\{(i,j): (i,j) \in H_q, (i,j) \in \bigcup_{i=q+1}^{k} B^l_i\right\}, q = 1, \ldots, k-1,$$

$$\bar{H}_k = H_k.$$

If

$$\max\left[\max_{\langle i,j\rangle \in \bar{H}_1} (U_1 + \theta_{ij} R^l_{1,ij}), \ldots, \max_{\langle i,j\rangle \in \bar{H}_k} (U_k + \theta_{ij} R^l_{k,ij})\right]$$
$$= \max_{\langle i,j\rangle \in \bar{H}_p} (U_p + \theta_{ij} R^l_{p,ij})$$
$$\equiv U_p + \theta_{st} R^h_{p,st} \text{ (say)},$$

then entry of $(s,t) \notin B^h_p$ into the set of basic cells, B^h_p, will yield a $(k+1)^{\text{th}}$ best basic feasible solution with $(k+1)^{\text{th}}$ best value in (P2) being $U_p + \theta_{st} R^h_{p,st}$.

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