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Theory and Methodology

A bilevel bottleneck programming problem

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Abstract

This paper studies a bilevel programming problem with linear constraints, and in which the objective functions at both levels are concave bottleneck functions which are to be minimized. The problem is a non-convex programming problem. It is shown that an optimal solution to the problem is attainable at an extreme point of the underlying region. The outer level objective function values are ranked in increasing order until a value is reached, one of the solutions corresponding to which is feasible for the problem. This solution is then the required global optimal solution.

Keywords: Non-linear programming; Non-convex programming; Bilevel programming; Bottleneck programming

1. Introduction

Intensive research is currently going on in the field of multilevel, and in particular, bilevel linear programming problems. Such problems occur in hierarchical administrative structures where there are decision makers at different levels of seniority, sharing a common pool of resources but controlling different sets of decision variables. Multilevel programming problems have a wide range of applications. Candler and Townsley [15] have suggested applications of multilevel programming in governmental problems involving issues such as the setting of penalties for illegal drug import, the fixing of import quotas and the development of transportation and communications infrastructure. Applications to strategic weapons exchange problems and to the distribution of federal budgets among states have been described respectively by Bracken et al. [14] and Cassidy et al. [16]. Anandalingam and Apprey [4] have given a new approach to conflict resolution based on multilevel mathematical programming and have illustrated it with a real world example of the Ganga water conflict problem between India and Bangladesh.

The bilevel programming problem can be thought of as a static version of the Stackelberg leader-follower game [31] in which a Stackelberg strategy is used by the leader (or the higher level decision maker), given the rational reaction of the follower (or the lower level decision maker). In a typical bilevel programming situation, the higher level decision maker is the central government or a central authority which sets policies, and the lower level decision makers are the state governments, industrial managers and the like, who work within the framework of these policies. The bilevel programming structure has been used

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to model problems concerning spatial competition [22], facility location [32], signal optimization [24] and traffic assignment [10]. Ben-Ayed et al. [9] have given a real-world bilevel programming model of the highway network design problem.

The bilevel linear programming problem:

(BLPP)

$$\max_X \; aX + bY$$

where Y solves

$$\max_Y \; (cX + dY/X) \qquad \text{(inner problem)}$$

$$\text{subject to} \quad AX + BY = b, \quad X, Y \geqslant 0,$$

is a non-convex programming problem [12] and is NP-hard [8].

Different solution methodologies have been proposed to solve (BLPP). Bialas et al [13] suggested the Parametric Complementary Pivot (PCP) algorithm to solve (BLPP), in which the inner problem is replaced by its K–T conditions and the outer objective function $aX + bY$ increased parametrically, taking care of the complementary slackness conditions. However, the algorithm contains certain inadequacies as has been pointed out by Ben-Ayed and Blair [8]. Bard [6] has suggested a Grid Search Algorithm (GSA) for global optimality of (BLPP). He first reduces (BLPP) to a semi-infinite linear programming problem and then applies Kuhn–Tucker theory to this resulting problem. But Haurie et al. [23] have given a counter example in which the algorithm does not converge to the optimal solution, and have discovered the errors in the formulation of the algorithm.

Candler and Townsley [15] and Bialas and Karwan [12] have suggested algorithms which find local optimal solutions to (BLPP), but these algorithms are unable to identify the global optimal solution. Ünlü [33] has used the relationship between the bilevel linear problem and the bicriteria linear problem to find an optimal solution to the former. The 'k-th best' algorithm of Bialas and Karwan [11] uses an extreme point search procedure to find a global optimal solution to (BLPP). In a technique suggested by Önal [28], the follower's problem is replaced by its equivalent Kuhn–Tucker conditions and the complementary slackness equations are moved to the leader's objective as a penalty function. The global optimal of the resulting quadratic programming problem is then reached by iterative application of a modified simplex algorithm. Anandalingam and White [3] and White and Anandalingam [34] have developed a solution methodology for the bilevel linear programming problem using a duality gap-penalty function format. An exact penalty function is shown to exist for obtaining its global optimal solution. It has also been shown that this methodology can be extended to solve certain non-linear bilevel problems. Shimizu and Aiyoshi [30] and Aiyoshi and Shimizu [1] have also used a penalty function approach for solving non-linear bilevel programming problems.

Besides the bilevel linear programming problem, the discrete bilevel programming problem, the mixed integer non-linear bilevel programming problem and the mixed integer linear bilevel programming problem have also been studied [7,17,26].

This paper introduces and studies the bilevel bottleneck programming problem (BBPP):

(P1 or BBPP)

$$\min_X \; F(X, Y)$$

where Y solves

(P2)

$$\min_Y \; (G(X, Y)/X)$$

$$\text{subject to} \quad AX + BY = b, \quad X, Y \geqslant 0,$$

and $F(X, Y)$ and $G(X, Y)$ are both concave bottleneck functions.

The bottleneck function was first studied by Hammer [21] in the context of the Time Minimizing Transportation Problem and also appears in the Time Minimizing Assignment Problem [19].

The general bottleneck linear programming problem has been studied by Akgül [2], Frieze

[18], Gupta et al. [20] and Seshan et al. [29]. Bansal et al. [5] have given a procedure for ranking of solutions of the bottleneck linear programming problem and also to find its alternate k-th best ($k \geq 1$) solutions. Minoux [25] has given many applications of the problem in which the objective function is the sum of a linear part and a bottleneck part.

A simple example of a practical application of the bilevel bottleneck programming problem being studied in this paper can be described as follows:

Consider a production set-up in which two categories of products, say, primary and secondary, are being produced. Some of the resources (e.g., raw material and machinery) are required for both kinds of products and their availability governed by linear constraints. In addition, the secondary products are also required as inputs for the primary ones, so that production of the latter can start only when production of the secondary products is complete. Clearly, it is the production plan of the primary products (represented by the variables in X) which is made first, and the plan for the secondary products (represented by the variables in Y) is made accordingly. If $F(X, Y)$ and $G(X, Y)$ are the times taken for the two phases of production respectively, and

$$AX + BY = b, \quad X \geq 0, \quad Y \geq 0,$$

are the given linear constraints, then the problem (BBPP) represents the mathematical model of such a set-up.

It has been proved in this paper that an optimal solution to (BBPP) is attainable at an extreme point of the region

$$S = \{(X, Y) \in \mathbb{R}^{N(1)} \times \mathbb{R}^{N(2)} \mid AX + BY = b, \quad X, Y \geq 0\},$$

and hence the extreme point search algorithm seems to be the most suitable one for finding a global optimal solution to (BBPP). The details of the algorithm have been presented in Section 4. The main results for developing the algorithm are established in Section 3 while the next section gives a description of the problem being studied.

2. Details of the problem

The bilevel bottleneck programming problem being studied in this paper is:

(P1)

$$\underset{X}{\text{Min}} \; F(X, Y)$$

where Y solves

(P2)

$$\underset{Y}{\text{Min}} \; (G(X, Y)/X)$$

subject to $(X, Y) \in S$,

where $S = \{(X, Y) \in \mathbb{R}^{N(1)} \times \mathbb{R}^{N(2)} \mid AX + BY = b, X, Y \geq 0\}$, where $A \in \mathbb{R}^{m \times N(1)}$, $B \in \mathbb{R}^{m \times N(2)}$, $b \in \mathbb{R}^m$, $N(1) + N(2) = n$, $\rho(A) = \rho(B) = m$, and S is assumed to be regular.

Define the index sets $I = \{1, \ldots, N(1)\}$ and $J = \{1, \ldots, N(2)\}$. The functions $F(X, Y)$ and $G(X, Y)$ are given as

$$F(X, Y) = \text{Max}\{f_i(x_i), i \in I; f_j'(y_j), j \in J\},$$

$$G(X, Y) = \text{Max}\{g_i(x_i), i \in I; g_j'(y_j), j \in J\},$$

where for all $i \in I$,

$$f_i(x_i) = \begin{cases} f_i & \text{if } x_i > 0 \; (f_i \in \mathbb{R}^+), \\ 0, & \text{otherwise,} \end{cases}$$

and $f_j'(y_j)(\forall j \in J)$, $g_i(x_i)(\forall i \in I)$ and $g_j'(y_j)(\forall j \in J)$ are similarly defined.

At this stage some remarks and observations on (P1), along with some points of comparison with (BLPP) would help in a better understanding of the problem being studied.

Some remarks and observations

1) The problems (P1) and (P2) are called the outer and inner problems respectively. The vectors X and Y are called policy and behavioral vectors respectively and the corresponding variables are called policy (or outer) and behavioral (or inner) variables [15]. (P1) and (P2) are also called the leader's and follower's problem respectively, and X and Y are correspondingly called the leader's and the follower's decision variables [31].

2) For a given $\bar{X} \in \mathbb{R}^{N(1)}$, the feasible region for (P2) is given by $S_{\bar{X}} = \{Y \in \mathbb{R}^{N(2)} \mid BY = b - A\bar{X},\ Y \geq 0\}$. Thus the inner problem for $\bar{X} \in \mathbb{R}^{N(1)}$ can also be written as $\text{Min}_{Y \in S_{\bar{X}}} G(\bar{X}, Y)$. Clearly, $S_{\bar{X}}$ is a convex region.

$S_{\bar{X}}$ will be non-empty if there exists a $\bar{Y} \in \mathbb{R}^{N(2)}$ such that $(\bar{X}, \bar{Y}) \in S$.

3) The set $\{X \in \mathbb{R}^{N(1)} \mid X \geq 0\}$ is called the policy space of (P1). An $\bar{X}$ belonging to the policy space of (P1) is called a feasible or an infeasible policy setting for (P1) depending on whether $S_{\bar{X}}$ is non-empty or empty.

The set $(X \in \mathbb{R}^{N(1)} \mid X \geq 0,\ S_X \neq \emptyset)$ is called the feasible policy space of (P1) and clearly it is the projection of S on the policy space. Also, the feasible policy space is a convex region. It may be noted that an optimal feasible solution of (P1) need not necessarily correspond to an extreme point of the feasible policy space.

4) The feasible region for (P1) is

$$S_1 = \left\{(\bar{X}, \bar{Y}) \in S \mid \bar{X} \text{ is a feasible policy setting and } \bar{Y} \text{ solves } \underset{Y \in S_{\bar{X}}}{\text{Min}}\, G(\bar{X}, Y)\right\}$$

$$= \left\{(\bar{X}, \bar{Y}) \in S \mid S_{\bar{X}} \neq \emptyset,\ G(\bar{X}, \bar{Y}) = \underset{Y \in S_{\bar{X}}}{\text{Min}}\, G(\bar{X}, Y)\right\}.$$

Thus the outer problem may be written as $\text{Min}_{(X,Y) \in S_1} F(X, Y)$. Clearly, if $S_{\bar{X}} \neq \emptyset$ and $\bar{Y}$ is a non-optimal feasible solution of the inner problem at $\bar{X}$, then $(\bar{X}, \bar{Y}) \notin S_1$.

5) S_1 is, in general, not a convex set (the same is true in the linear case also [11]). S_1 is a union of disjoint convex sets, all of which are not necessarily closed. Also, S_1 is connected, in the sense that between any two points of S_1 there exists a piecewise linear continuous path lying entirely in S_1. The following example illustrates the observations made here:

$$\underset{X}{\text{Min}}\, F(X, Y)$$

where Y solves

$$\underset{Y}{\text{Min}}\, (G(X, Y)/X)$$

$$\text{subject to}\quad \begin{aligned} &x_1 + 4y_1 - y_2 = 16,\\ &x_1 + 4y_1 + y_3 = 32,\\ &-x_1 + y_1 + y_4 = 3,\\ &x_1 + 2y_1 + y_5 = 22,\\ &x_1 - y_1 + y_6 = 10,\\ &x_1 \geq 0, \quad y_j \geq 0, \quad j = 1, \ldots, 6, \end{aligned}$$

$f_1 = 14$, $f_1' = 16$, $f_2' = 31$, $f_3' = 29$, $f_4' = 20$, $f_5' = 35$, $f_6' = 26$.

$g_1 = 7$, $g_1' = 13$, $g_2' = 24$, $g_3' = 30$, $g_4' = 28$, $g_5' = 20$, $g_6' = 17$.

The two open, thatched regions A_1 and A_2 (not including the dotted boundaries) in Fig. 1, along with A_3 constitute the set S_1 of feasible solutions of (P1).

6) The 'weak convexity like property of S_1', which holds in the linear case [11], may not hold in this case.

7) An extreme point of S_1 is an extreme point of S.

8) Uniqueness of optimal solution of the inner problem is not essential. Unlike bilevel linear programming, the correspondence between the set S_1 and the feasible policy space may not be one–one.

9) Clearly, if $(\bar{X}, \bar{Y}) \in \text{Rel.Int}(S)$, the set of points in the relative interior of S, then $\bar{Y} \in \text{Rel.Int}(S_{\bar{X}})$. Thus if $\bar{Y}$ is an extreme point of $S_{\bar{X}}$, then $(\bar{X}, \bar{Y}) \notin \text{Rel.Int}(S)$, and hence $(\bar{X}, \bar{Y}) \notin \text{Int}(S)$, the set of interior points of S.

10) The set $S_{\bar{X}}^* = \{Y \in \mathbb{R}^{N(2)} \mid Y$ solves $\text{Min}_{Y \in S_{\bar{X}}} G(\bar{X}, Y)\}$ is a convext set. For a specified $\bar{X} \in \mathbb{R}^{N(1)}$, the minimum values of $F(\bar{X}, Y)$ and $G(\bar{X}, Y)$ are attainable at extreme points of $S_{\bar{X}}^*$ [5].

11) Every feasible solution of (P1) is a feasible solution of the linear bottleneck programming problem $\text{Min}_{(X,Y) \in S} F(X, Y)$. The optimal value of $F(X, Y)$ in $\text{Min}_{(X,Y) \in S} F(X, Y)$ is, in general, smaller than the optimal value of $F(X, Y)$ in (P1). If $(\bar{X}, \bar{Y})$ is an optimal basic feasible solution of $\text{Min}_{(X,Y) \in S} F(X, Y)$ with $\bar{Y}$ being an optimal basic feasible solution of $\text{Min}_{Y \in S_{\bar{X}}} G(\bar{X}, Y)$,

then $(\bar{X}, \bar{Y})$ is an optimal feasible solution of (P1).

Assumptions

The following assumptions have been made throughout this paper:

(i) $N(2) > m$, and

(ii) $g'_j > \max_{i \in I}\{g_i\}$ $\forall j \in J$.

If $N(2) \leqslant m$, then for any $\bar{X} \in \mathbb{R}^{N(1)}$, the system $BY = b - A\bar{X}$, $Y \geq 0$ has atmost one solution, due to which the inner problem is no longer meaningful. The second condition ensures that for any $\bar{X} \in \mathbb{R}^{N(1)}$, the bottleneck value of $G(\bar{X}, Y)$ is controlled by Y and so $G(\bar{X}, Y)$ does not have the same value on the entire feasible region $S_{\bar{X}}$ of the corresponding inner problem. In other words, the policy variables control only the feasibility of the inner problem and not its optimality.

Thus both the assumptions made here are necessary to retain the essential character of the bilevel problem being studied.

3. Some results

Theorem 1. *If there exists a feasible point for* (P1) *then there exists a feasible extreme point.*

Proof. Let $(\bar{X}, \bar{Y})$ be a feasible point for (P1). Then, $\bar{Y}$ solves $\text{Min}_{Y \in S_{\bar{X}}} G(\bar{X}, Y)$, which means that $\bar{Y}$ is an extreme point of $S_{\bar{X}}$ [5]. Therefore $(\bar{X}, \bar{Y}) \notin \text{Int}(S)$.

Suppose now that $(\bar{X}, \bar{Y})$ is not an extreme point of S, so that it is on an edge or facet of S [27]. Therefore, there exist extreme points (X^k, Y^k), $k = 1, \ldots, p$, of S lying on that edge or facet such that

$$(\bar{X}, \bar{Y}) = \sum_{k=1}^{p} \lambda_k (X^k, Y^k),$$

$$\sum_{k=1}^{p} \lambda_k = 1, \quad \lambda_k > 0 \ \forall k, \tag{1}$$

By definition of the bottleneck function G and as $\lambda_k > 0, \forall k$,

$$G(\bar{X}, \bar{Y}) = \underset{1 \leqslant k \leqslant p}{\text{Max}} \ G(X^k, Y^k) \tag{2}$$

Assume that none of the points (X^k, Y^k), $k = 1, \ldots, p$, is feasible for (P1). This implies, Y^k is not optimal for $\text{Min}_{Y \in S_{X^k}} G(X^k, Y)$ for each k.

Let $\hat{Y}^k \in S_{X^k} (k = 1, \ldots, p)$ be an optimal solution of $\text{Min}_{Y \in S_{X^k}} G(X^k, Y)$. Then

$$G(X^k, \hat{Y}^k) < G(X^k, Y^k) \quad \forall k = 1, \ldots, p \tag{3}$$

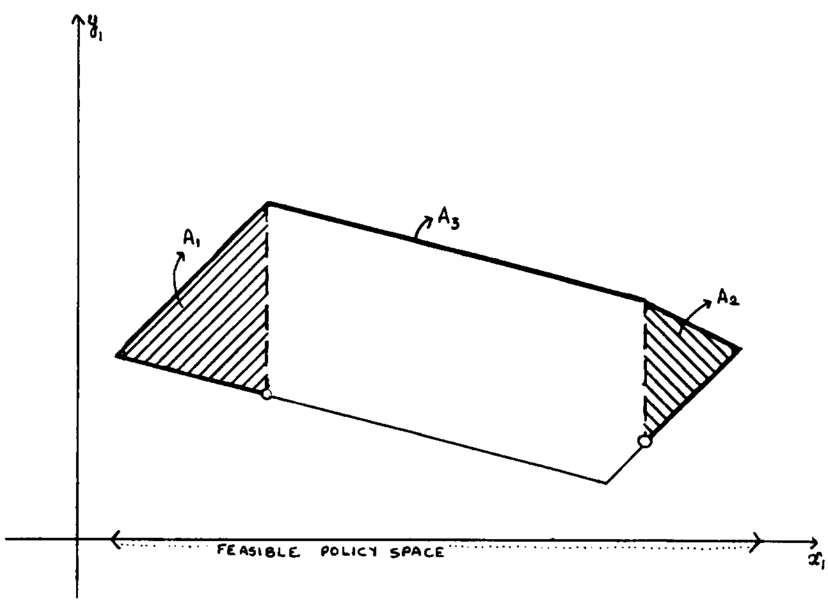

Fig. 1.

As $\hat{Y}^k \in S_{X^k}$, therefore $(X^k, \hat{Y}^k) \in S$, $\forall k = 1,\ldots,p$. S being convex, $\sum_{k=1}^{p} \lambda_k (X^k, \hat{Y}^k) \in S$, that is

$$(\bar{X}, \hat{Y}) \in S \quad \text{where } \hat{Y} = \sum_{k=1}^{p} \lambda_k \hat{Y}^k.$$

Therefore, $\hat{Y} \in S_{\bar{X}}$.

But $\bar{Y}$ is an optimal solution for $\text{Min}_{Y \in S_{\bar{X}}} G(\bar{X}, Y)$, since $(\bar{X}, \bar{Y})$ is feasible for (P1). Therefore,

$$G(\bar{X}, \bar{Y}) \le G(\bar{X}, \hat{Y}). \tag{4}$$

Now,

$$\begin{aligned} G(\bar{X}, \hat{Y}) &= \underset{1 \leqslant k \leqslant p}{\text{Max}}\, G(X^k, \hat{Y}^k) \\ &\qquad \text{since } (\bar{X}, \hat{Y}) = \sum_{k=1}^{p} \lambda_k (X^k, \hat{Y}^k) \\ &< \underset{1 \leqslant k \leqslant p}{\text{Max}}\, G(X^k, Y^k) \quad \text{from (3)} \\ &= G(\bar{X}, \bar{Y}), \qquad \text{from (2)} \end{aligned} \tag{5}$$

Eqs. (4) and (5) give a contradiction. Therefore the assumption that none of the points $(X^k, Y^k)(1 \le k \le p)$ is feasible for (P1) is wrong. Hence if $(\bar{X}, \bar{Y})$ is feasible for (P1), there exists at least one extreme point of S such that it is feasible for (P1).

Theorem 2. *The optimal value of $F(X, Y)$ in* (P1) *is attainable at an extreme point of S.*

Proof. By definition of the bottleneck function, for each pair of points $(X, Y) \in \text{Rel.Int}(S)$ and $(X', Y') \notin \text{Rel.Int}(S)$, $F(X', Y') \le F(X, Y)$. Therefore, if an optimal feasible solution $(\bar{X}, \bar{Y})$ (say) of (P1) is not an extreme point of S, it must lie on an edge or a facet of S. Now by Theorem 1, there exists an extreme point, say $(\hat{X}, \hat{Y})$ of S, lying on that edge or facet, which is feasible for (P1). Now, $(\bar{X}, \bar{Y})$ can be expressed as

$$(\bar{X}, \bar{Y}) = \lambda(\hat{X}, \hat{Y}) + (1-\lambda)(X, Y), \quad 0 < \lambda < 1,$$

where (X, Y) is any other point on that edge or facet [27].

By definition of the bottleneck function F and since $0 < \lambda < 1$, $F(\bar{X}, \bar{Y}) \ge F(\hat{X}, \hat{Y})$. But since $(\bar{X}, \bar{Y})$ is an optimal solution to (P1) and $(\hat{X}, \hat{Y})$ is a feasible solution for the same,

$$F(\bar{X}, \bar{Y}) \ngtr F(\hat{X}, \hat{Y}).$$

Hence, $F(\bar{X}, \bar{Y}) = F(\hat{X}, \hat{Y})$, which implies that the optimal value of $F(X, Y)$ in (P1) is attainable at an extreme point of S.

Theorem 3 (A feasibility test for (P1)). *Let $(\bar{X}, \bar{Y}) \in S$. Then $(\bar{X}, \bar{Y})$ is feasible for* (P1) *if the optimal value of $Z(Y)$ is non-zero and finite in the linear programming problem*

LP($\bar{G}/\bar{X}$)

$$\underset{S_{\bar{X}}}{\text{Min}}\; Z(Y) = \sum_{j=1}^{N(2)} c_j y_j,$$

where

$$c_j = \begin{cases} 0, & g_j' < \bar{G}, \\ 1, & g_j' = \bar{G}, \\ \infty, & g_j' > \bar{G}, \end{cases} \qquad j = 1,\ldots,N(2),$$

and $\bar{G} = G(\bar{X}, \bar{Y})$.

Proof. $(\bar{X}, \bar{Y})$ is feasible for (P1) if $\bar{Y}$ solves (P2) for $\bar{X}$. Assume that $\bar{Y}$ is not an optimal solution for (P2). Then there exists a $\hat{Y} \in \mathbb{R}^{N(2)}$ such that $\hat{Y} \in S_{\bar{X}}$ and $G(\bar{X}, \hat{Y}) < G(\bar{X}, \bar{Y}) = \bar{G}$. This implies $Z(\hat{Y}) = 0$, which contradicts the fact that the optimal value of $Z(Y)$ in LP($\bar{G}/\bar{X}$) is non-zero.

Theorem 4. *Any k-th best solution ($k \ge 1$) of the bottleneck linear programming problem $\text{Min}_{W \in S} F(W)$ is attainable at an extreme point of S.*

Proof. (For convenience of notation, (X, Y) has been replaced by W). Let F_k be the k-th best value of $F(W)$ in the problem $\text{Min}_{W \in S} F(W)$. Let $F(\bar{W}) = F_k$ and suppose that $\bar{W}$ is not an extreme point of S. Then there exist extreme points $W_1, W_2, \ldots, W_\ell$ of S such that

$$\bar{W} = \sum_{i=1}^{l} \lambda_i W_i, \qquad \sum_{i=1}^{\ell} \lambda_i = 1, \quad 0 < \lambda_i < 1 \;\; \forall i.$$

By definition of the bottleneck function F and since $0 < \lambda_i < 1$, $\forall i$,

$$F_k = F(\overline{W}) = \underset{1 \leqslant i \leqslant l}{\text{Max}} F(W_i)$$
$$= F(W_s) \quad (\text{say}) \quad (1 \leqslant s \leqslant l).$$

Therefore any k^{th} best solution of $\text{Min}_{W \in S} F(W)$ is attainable at an extreme point of S.

4. Algorithm

It has now been proved that an optimal solution of (P1) is attainable at an extreme point of S (Theorem 2). Also, any k^{th}- best value of $F(X, Y)$ in $\text{Min}_{(X,Y) \in S} F(X, Y)$ is also attainable at an extreme point of S (Theorem 4). Therefore, to find the optimal solution to (P1), the extreme points of S are scanned by ranking the solutions of $\text{Min}_S F(X, Y)$. The successive solutions are tested for feasibility for (P1) (Theorem 3) and the algorithm stops the moment feasibility is attained. Since S is regular, it has a finite number of extreme points, and so the the algorithm is bound to terminate in a finite number of steps.

Step 1. Set $i = 1$, $T = \emptyset$.

Step 2. Find the set S^i of ith best basic feasible solutions of $\text{Min}_{(X,Y) \in S} F(X, Y)$ [5].

Step 3. If $S^i \setminus T = \emptyset$, set $i = i + 1$, $T = \emptyset$ and return to Step 2.
Choose some $(\overline{X}, \overline{Y}) \in S^i \setminus T$. Let $G(\overline{X}, \overline{Y}) = \overline{G}$.
Solve the linear programming problem $\text{LP}(\overline{G}/\overline{X})$ as defined in Theorem 3. Let its optimal solution be $\hat{Y}$. If $Z(\hat{Y})$ is non-zero and finite, go to Step 4.
If $Z(\hat{Y}) = 0$, set $T = T \cup \{(\overline{X}, \overline{Y})\}$ and return to Step 3.

Step 4. $(\overline{X}, \overline{Y})$ is an optimal solution to (P1).

Example. Consider the problem (P1) where the set S is given by the constraints

$$x_1 + 2x_2 + y_1 + 2y_2 + 3y_4 = 6,$$
$$3x_1 + x_2 + 2y_1 + y_3 + 2y_4 = 5,$$
$$x_1 + x_2 + y_1 + y_3 + y_4 = 3,$$
$$x_1,\ x_2 \geq 0,\ y_1,\ y_2,\ y_3,\ y_4 \geq 0,$$

and $f_1 = 10$, $f_2 = 6$, $f_1' = 3$, $f_2' = 15$, $f_3' = 21$, $f_4' = 4$.
$g_1 = 3$, $g_2 = 5$, $g_1' = 9$, $g_2' = 22$, $g_3' = 7$, $g_4' = 30$.
The set S^1 of optimal basic feasible solutions of $\text{Min}_S F(X, Y)$ is $S^1 = \{(0, 1, 1, 0, 0, 1)\}$. $\therefore$ $(\overline{X}, \overline{Y}) = (0, 1, 1, 0, 0, 1)$ with $G(\overline{X}, \overline{Y}) = 30 = \overline{G}$.
The optimal value of the objective function in the linear programming problem $\text{LP}(\overline{G} = 30/\overline{X})$ is zero.
So (0, 1, 1, 0, 0, 1) is not feasible for (P1).
Now $S^2 = \{(\frac{2}{3}, \frac{5}{3}, 0, 0, 0, \frac{2}{3})\}$.
$\therefore$ $(\overline{X}, \overline{Y}) = (\frac{2}{3}, \frac{5}{3}, 0, 0, 0, \frac{2}{3})$ with $G(\overline{X}, \overline{Y}) = 30 = \overline{G}$.
The optimal value of the objective function in the linear programming problem $\text{LP}(\overline{G} = 30/\overline{X})$ is zero.
So $(\frac{2}{3}, \frac{5}{3}, 0, 0, 0, \frac{2}{3})$ is also not feasible for (P1).

Next, $S^3 = \{(1, 2, 0, \frac{1}{2}, 0, 0),\ (0, 1, 2, 1, 0, 0)\}$. Choose $(\overline{X}, \overline{Y}) = (1, 2, 0, \frac{1}{2}, 0, 0)$. $G(\overline{X}, \overline{Y}) = 22$.
The optimal value of the objective function in $\text{LP}(\overline{G} = 22/\overline{X})$ is non-zero and finite.
Hence $(1, 2, 0, \frac{1}{2}, 0, 0)$ is feasible and optimal for (P1).
Note: The point (0, 1, 2, 1, 0, 0) is also an optimal feasible solution for (P1).

5. Concluding remarks

Suppose $(X^*, Y^*) \in S^k$ is an optimal solution of (P1). To find alternate optimal solutions to (P1) each of the remaining points in S^k should be tested for feasibility and those which are feasible for (P1) are alternate optimal solutions.

If a second best solution of (P1) is to be found, test the points of S^{k+1} for feasibility for (P1). If there is some $(X, Y) \in S^{k+1}$ which is feasible for (P1),then (X, Y) is a second best solution of (P1). If none of the points in S^{k+1} is feasible for (P1), go to S^{k+2}, and so on. Continuing in this way, the 3rd, 4th,... best solutions of (P1) can be found.

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