Two Stage Interval Time Minimizing Transportation Problem

Sonia[a], Rita Malhotra[b] and M.C. Puri[c]

Abstract

A Two Stage Interval Time Minimizing Transportation Problem, where total availability of a homogeneous product at various sources is known to lie in a specified interval, is studied in the present paper. In the first stage, the sources ship all of their on-hand material to the demand points, while a second-stage delivery covers the demand that is not fulfilled in the first shipment. In each stage, the objective is to minimize the shipment time, and the overall goal is to find a solution that minimizes the sum of the first- and second-stage shipment times. A polynomial time algorithm is proposed to solve the problem to optimality, where at various steps of the algorithm lexicographic optimal solutions of restricted versions of a related standard time minimizing transportation problem are examined and finally the global optimal solution is determined.

Keywords: Combinatorial Optimization, Non-Convex Optimization, Time Minimizing Transportation Problem, Global Optimization.

Introduction

Wide ranging literature is available to study the cost minimizing transportation problem (CMTP). If $I=\{1,2,...,m\}$ is the index set of m sources, $J=\{1,2,...,n\}$ of n destinations, $c_{ij},(i,j)\in I\times J$ the per unit shipment cost from the source i to the destination j, $a_i,\ i\in I$ the availability of a homogeneous product at the source i and $b_j,\ j\in J$ the demand of the same at the destination j, then the standard CMTP is modeled as:

$$\min \sum_{(i,j)\in I\times J} c_{ij}x_{ij}$$

subject to

$$\left.\begin{aligned} &\sum_{j\in J} x_{ij} = a_i, i\in I\\ &\sum_{i\in I} x_{ij} = b_j, j\in J\\ &x_{ij}\geq 0, \forall (i,j)\in I\times J \end{aligned}\right\} \quad \text{... (1.0)}$$

where, $x_{ij},(i,j)\in I\times J$ denotes the quantity shipped from the source i to the destination j. The best-known strongly polynomial time algorithm for CMTP is of order $O(m\log n(m+n\log n))$ (Orlin, 1988).

The time minimizing transportation problem (TMTP) is another important class of transportation problems in terms of its widespread applications. If $t_{ij}(x_{ij})$, $(i,j)\in I\times J$, the shipment time from the source i to the destination j, is defined as:

$$\left.\begin{aligned} &t_{ij}(x_{ij}) = t_{ij}(\geq 0), && \text{if } x_{ij}>0\\ &t_{ij}(x_{ij}) = 0, && \text{if } x_{ij}=0 \end{aligned}\right\}$$

and shipment from sources to destinations is done in parallel, then the mathematical model for the standard TMTP is:

$$\min[T(X)=\max_{I\times J}(t_{ij}(x_{ij})]$$

where $X=(x_{ij})$ satisfies (1.0). It may be noted that shipment time from the i^{th} source to the j^{th} destination does not depend upon the volume of the shipment. Clearly, $t_{ij}(x_{ij})$ is a concave function. $T(X)$ is the shipment time for a feasible schedule X and is known to be a concave function (Bansal and Puri, 1980). Hence, standard TMTP is a concave minimization problem (CMP).

In literature conventional TMTP has been studied by various authors (Ahuja, 1986), (Bhatia, Kanti Swarup and Puri, 1976),

[a] Indian Institute of Technology, Delhi, India (Email: sonia_iit@rediffmail.com)
[b] Kamala Nehru College, Delhi University, Delhi, India
[c] Indian Institute of Technology, Delhi, India (Email: mcpuri@maths.iitd.ernet.in)

(Garfinkel and Rao, 1971), (Hammer, 1969) and TMTPs with mixed constraints and flow constraints have been studied by Khanna et al. (Khanna, Bakshi and Puri, 1981), (Khanna and Puri, 1983). An optimal solution of TMTP can be obtained by finding its lexicographic optimal solution (LOS) (Satya Prakash, 1982). A lexicographic optimal solution of TMTP is one in which not only the shipment on the longest duration routes is minimized but shipments on all other routes of various durations are also minimized where routes mean various source-destination links. For obtaining an LOS, the set of transportation times on various routes is partitioned into a number of disjoint sets, $B_k, k = 1,2,...,s$, where, $B_k = \{(i, j) \in I \times J : t_{ij} = T^k\}$ and $T^{j+1} > T^j$ $\forall\, j = 1,2,\ldots,s-1$. Positive weights, say $\lambda_k,\ k = 1,2,\ldots,s$, are attached to these sets where, $\lambda_{j+1} >> \lambda_j\ \forall\, j = 1,2,\ldots,s-1$. This yields a standard CMTP: $\min \sum_{k=1}^{s} \lambda_k \left(\sum_{(i,j)\in B_k} x_{ij} \right)$ where, $X = (x_{ij})$ belongs to the transportation polytope over which the original TMTP is being studied. An optimal feasible solution of this CMTP yields a LOS of TMTP. Thus, a TMTP is also solvable in polynomial time. The positive weights $\lambda_j, j = 1,2,\ldots,s$ can be determined as described by Sherali (Sherali, 1982) and Mazzola (Mazzola, 1993).

There are many more problems where it becomes necessary to study a TMTP wherein the products are shipped to the destinations in two stages. Consider for example, the production of maintenance-free-sealed industrial batteries. Production is a continuous process depending on the available resources. However each battery has a certain shelf life and batteries need to be periodically re-charged, else the whole lot becomes dead resulting in the loss of the finished goods. Often due to lack of re-charging facilities on the production floor, each batch of manufactured batteries is transported immediately to the demand points; this corresponds to the first stage. In the second stage enough maintenance-free-sealed batteries from the sources are shipped in order to satisfy the industrial users' demands at the destinations. Shipment is done in such a way as to minimize the overall transportation time. Such situations motivated the study of a two-stage interval time minimizing transportation problem where the total availability of the product at the sources lies in specified intervals. In both the stages, transportation of the product from the sources to the destinations is done in parallel.

The current problem may be viewed as a parametric optimization problem (Gal, 1979) in which availability at the source $i(i \in I)$ varies in the interval $[a_i, a'_i]$. But as the present problem consists of two stages, second stage being dependent upon first stage, the parametric approach will not be of much advantage.

The current problem, in spite of interval sources' constraints, is very much different from interval linear programming because of the sum of two dependent concave functions in the objective function. Therefore, solution strategies available for solving interval linear programming problem (Charnes and Granot, 1976), (Robers and Israel, 1970) will not only be computationally expensive but of no use as well. Similarly inexact optimization techniques (Amaya and Ghellinck, 1997), (Soyster, 1973) will not be of much use.

The Two Stage Interval TMTP is shown to be related to an ordinary interval TMTP, which is further shown to be equivalent to a standard TMTP. Feasible solutions of the Stage-I and Stage-II problems are derived from a feasible solution of this standard TMTP. Due to the dependence of Stage-II on Stage-I, special types of solutions viz. lexicographic optimal solutions (LOS) are investigated. First, Stage-II shipment time is being controlled by solving the various restricted versions of this standard TMTP in which some routes pertaining to Stage-II are abandoned. In the Stage-I and Stage-II shipment time pairs thus obtained successively, Stage-II shipment time strictly decreases and Stage-I shipment time strictly increases. Similarly on the other hand, Stage-I shipment time is controlled by solving other restricted versions of the same standard TMTP in which some routes pertaining to Stage-I are abandoned. In the pairs, thus generated, Stage-I shipment time keeps on decreasing and Stage-II shipment time keeps on increasing. In all the pairs of Stage-I and Stage-II shipment times, the shipment time of one stage is the minimum corresponding to the other.

As during the algorithm a finite number $(\leq 4(s-r)-2)$ of CMTPs are to be solved, it follows that the proposed algorithm is also a polynomial time algorithm, where s is the number of partitions of transportation times on various routes and r is the position in the ordering of shipment times on various routes (arranged in the descending order) of the overall minimum shipment time for the standard TMTP related to Two Stage Interval TMTP. This minimum shipment time is yielded by an LOS of this standard TMTP.

Theoretical development of the problem is presented in the next section followed by the development of the algorithm. Concluding remarks are given towards the end after a numerical illustration.

Theoretical Development

<u>Mathematical Formulation of Two Stage Interval TMTP</u>

Let a_i and $a'_i, i \in I$ denote respectively the minimum and maximum availability of a homogeneous product at the source i and $b_j, j \in J$ the demand of the same at the destination j, where $\sum_I a_i < \sum_J b_j < \sum_I a'_i$.

In the first stage of the Two Stage Interval TMTP the quantity $a_i (< a'_i)$ is shipped from each source $i, i \in I$ and after its completion, enough quantity of the product is dispatched in the second stage so as to exactly satisfy the demand b_j at the destination $j, j \in J$.

The Stage-I problem is thus formulated as:

$$\min_{Y=(y_{ij})\in S'}\left[\max_{I\times J}\left(t_{ij}(y_{ij})\right)\right]=\min_{Y\in S'}\left[T_1(Y)\right]$$

where, the set S' is given by

$$S':\begin{cases}\sum_{j\in J} y_{ij} = a_i, i \in I\\ \sum_{i\in I} y_{ij} \leq b_j, j \in J\\ y_{ij} \geq 0, \forall (i,j) \in I\times J\end{cases}$$

Corresponding to a feasible solution $Y=(y_{ij})$ of the Stage-I problem, let $S'(Y)$ be the set of feasible solutions of the Stage-II problem which is stated as:

$$\min_{Z\in S'(Y)}\left[\max_{I\times J}\left(t_{ij}(z_{ij})\right)\right]=\min_{Z\in S'(Y)}\left[T_2(Z)\right]$$

where,

$$S'(Y):\begin{cases}\sum_{j\in J} z_{ij} \leq a'_i - a_i, i \in I\\ \sum_{i\in I} z_{ij} = b_j - b'_j, j \in J\\ z_{ij} \geq 0, \forall (i,j) \in I\times J\end{cases}$$

and $b'_j = \sum_{i\in I} y_{ij}, j \in J.$

Thus, the Two Stage Interval Time Minimizing Transportation Problem can be stated as

$$\min_{Y\in S'}\left[T_1(Y)+\left(\min_{Z\in S'(Y)} T_2(Z)\right)\right] \qquad (P)$$

As shipment times in Stage-I and Stage-II are concave functions, the Two Stage Interval TMTP aims at minimizing a concave function over a polytope. Hence (P) is also a concave minimization problem. As global minimizer of a CMP over a polytope is attainable at an extreme point of the polytope, it is desirable to investigate only its extreme points.

Closely related to the problem (P) is the interval time minimizing transportation problem (P_α) defined as

$$\min_{X\in S}\left[T(X)\right]=\min_{X\in S}\left[\max_{I\times J}\left(t_{ij}(x_{ij})\right)\right] \qquad (P_\alpha)$$

where,

$$S:\begin{cases}a_i \leq \sum_{j\in J} x_{ij} \leq a'_i, i \in I\\ \sum_{i\in I} x_{ij} = b_j, j \in J\\ x_{ij} \geq 0, \forall (i,j) \in I\times J\end{cases}$$

Clearly a feasible solution of the problem (P) provides a feasible solution to the problem (P_α) and conversely.

The standard time minimizing transportation problem (P_β) associated with (P_α) is defined as

$$\min_{X\in \hat{S}}\left[\hat{T}(X)\right]=\min_{X=(x_{ij})\in \hat{S}}\left[\max_{\hat{I}\times \hat{J}}\left(\hat{t}_{ij}(x_{ij})\right)\right] \qquad (P_\beta)$$

where,

$$\hat{S}:\begin{cases}\sum_{j\in\hat{J}} x_{ij} = \hat{a}_i, i\in\hat{I}\\ \sum_{i\in\hat{I}} x_{ij} = \hat{b}_j, j\in\hat{J}\\ \quad x_{ij}\geq 0, \forall (i,j)\in\hat{I}\times\hat{J}\end{cases}$$

where,

$\hat{I} = \{1,2,\ldots m, m+1,\ldots,2m\}$

$\hat{J} = J\bigcup\{n+1\}$

$\hat{a}_i = a_i, i=1,2,\ldots m$

$\hat{a}_{m+i} = a'_i - a_i, i=1,2,\ldots m$

$\hat{b}_j = b_j, j\in J$

$\hat{b}_{n+1} = \sum_{i\in I} a'_i - \sum_{j\in J} b_j$

$\hat{t}_{ij} = t_{ij}, i=1,2,\ldots m, j\in J$

$\hat{t}_{m+i,j} = t_{ij}, i=1,2,\ldots m, j\in J$

$\hat{t}_{i,n+1} = M, i=1,2,\ldots m, M >> 0$

$\hat{t}_{m+i,n+1} = 0, i=1,2,\ldots m$

An LOS of the problem (P_β) will provide the overall minimum shipment time for (P_β).

Let $\min_{X\in\hat{S}}[\hat{T}(X)] = T^r \equiv T^{r-p_0}\,(p_0 = 0)$.

A feasible solution $X=(x_{ij})$ of the problem (P_β) is said to be an M-feasible solution (MFS) if $x_{ij} = 0\,\forall (i,j): t_{ij} = M$.

It may be noted that if $X=(x_{ij})_{\hat{I}\times\hat{J}}$ is an M-feasible solution of problem (P_β), then $Y = (y_{ij})_{I\times J}$ and $Z=(z_{ij})_{I\times J}$ are respectively feasible solutions of Stage-I and Stage-II problems where, $y_{ij} = x_{ij}\,\forall (i,j)\in I\times J$ and $z_{ij} = x_{m+i,j}\,\forall (i.j)\in I\times J$. Further if $\min_{Z\in S'(Y)}[T_2(Z)] = T_2(\hat{Z})$, then a feasible solution of the problem (P) consists of the feasible solution Y for Stage-I and the feasible solution $\hat{Z}$ for Stage-II.

The Next two theorems establish the equivalence between the problem (P_α) and the problem (P_β) over the set of its M-feasible solutions.

Theorem 1 An M-feasible solution of the problem (P_β) corresponds to a feasible solution of the problem (P_α) and vice versa.

Proof Let $Y=(y_{ij})$ be an M-feasible solution of the problem (P_β). For each $i\in I$, define $x_{ij} = y_{ij} + y_{m+i,j}\,\forall j\in J$. It can be easily established that $X=(x_{ij})$ is a feasible solution of the problem (P_α).

Conversely, let $X=(x_{ij}),(i,j)\in I\times J$ be a feasible solution of the problem (P_α).

Let $\sum_{j\in J} x_{ij} = \overline{a}_i, i\in I$. Clearly $a_i \leq \overline{a}_i \leq a'_i$.

Therefore,

$$\sum_{j\in J} x_{ij} = \overline{a}_i, i\in I$$
$$\sum_{i\in I} x_{ij} = b_j, j\in J$$
$$x_{ij}\geq 0, \forall (i,j)\in I\times J$$

Clearly, $\sum_{i\in I}\overline{a}_i = \sum_{j\in J} b_j$

Consider the following imbalanced TMTP:

$$\min T(Z) = \min\left[\max_{I\times J}(t_{ij}(z_{ij})\right]$$

subject to

$$\sum_{j\in J} z_{ij} = a_i, i\in I$$
$$\sum_{i\in I} z_{ij} \leq b_j, j\in J$$
$$z_{ij}\geq 0, \forall (i,j)\in I\times J$$

Let $Z=(z_{ij})$ be an optimal feasible solution of this problem and let $\sum_{i\in I} z_{ij} = \overline{b}_j, j\in J$. Now, set

$$\left.\begin{array}{l} y_{ij} = z_{ij}, (i,j)\in I\times J\\ y_{i,n+1} = 0, \forall i\in I\end{array}\right\} \qquad (1.1)$$

Next, consider the balanced TMTP defined as follows:

$$\min T(W) = \min\left[\max_{I\times J}(t_{ij}(w_{ij})\right]$$

subject to,

$$\sum_{j\in J} w_{ij} = \bar{a}_i - a_i, i \in I$$

$$\sum_{i\in I} w_{ij} = b_j - \bar{b}_j, j \in J$$

$$w_{ij} \geq 0, \forall (i,j) \in I \times J$$

If $W = (w_{ij})$ is an optimal solution of the above problem, then set

$$\left.\begin{aligned} y_{m+i,j} &= w_{ij}, (i,j) \in I \times J \\ y_{m+i,n+1} &= a'_i - \bar{a}_i, i \in I \end{aligned}\right\} \quad (1.2)$$

It can be easily seen that $Y = (y_{ij}), (i,j) \in \hat{I} \times \hat{J}$ as defined by (1.1) and (1.2) above, is an M-feasible solution of the problem (P_β).

Hence the result. ■

Theorem 2 The value of the objective function of the problem (P_β) at an M-feasible solution is same as the value of objective function of the problem (P_α) at the corresponding feasible solution.

Proof Let $Y = (y_{ij})$ be an M-feasible solution of the problem (P_β) with the value of its objective function as T_β. Let $X = (x_{ij})$ be the corresponding feasible solution of the problem (P_α) giving T_α as the value of its objective function. Then,

$$T_\beta = \max_{\hat{I}\times\hat{J}} \left(\hat{t}_{ij}(y_{ij})\right)$$

$$= \max \begin{Bmatrix} \max_{I\times J} \left(t_{ij}(y_{ij})\right), \\ \max_{I} \left(t_{i,n+1}(y_{i,n+1})\right), \\ \max_{I\times J} \left(t_{m+i,j}(y_{m+i,j})\right), \\ \max_{I} \left(t_{m+i,n+1}(y_{m+i,n+1})\right) \end{Bmatrix}$$

$$= \max\left\{ \max_{I\times J} \left(t_{ij}(y_{ij})\right), \max_{I\times J}\left(t_{m+i,j}(y_{m+i,j})\right)\right\}$$

$$= \max_{I\times J} \left\{ t_{ij}(x_{ij}) \right\} \text{ since } x_{ij} = y_{ij} + y_{m+i,j} \text{ and } t_{m+i,j} = t_{ij}, (i,j) \in I \times J$$

$$= T_\alpha$$

■

<u>Solution Strategy for the Two Stage Interval TMTP</u>

The solution strategy for the Two Stage Interval TMTP depends upon the following types of standard TMTPs and CMTPs.

The possibility of Stage-II shipment time less than the on hand shipment time, say T^{l-1}, is examined by studying the time minimizing transportation problem $P_{L\beta}(T^{l-1})$ derived from the problem (P_β) by abandoning the routes $(m+i,j)$ for which $t_{m+i,j} \geq T^{l-1}, (i,j) \in I \times J$ i.e., setting $t_{m+i,j} = M(>> 0), (i,j) \in I \times J$ for which $t_{m+i,j} \geq T^{l-1}$. If LOS of $P_{L\beta}(T^{l-1})$ is an MFS, then the new Stage-I shipment time is more than the on hand Stage-I shipment time and the corresponding Stage-II shipment time is less than the on hand Stage-II shipment time.

Suppose LOS of $P_{L\beta}(T^{l-1})$ is M-feasible and let Stage-I and Stage-II shipment times corresponding to this LOS be T^k and T^l respectively. To find the least possible Stage-II shipment time corresponding to the Stage-I shipment time T^k, the following cost minimizing transportation problem (call it $CP_{L\beta}(T^k, T^l)$) is solved.

$$\min_{\hat{S}} \sum_{I\times J} c_{ij} x_{ij} \qquad \ldots CP_{L\beta}(T^k, T^l)$$

where,

$$\begin{aligned} c_{ij} &= M, \quad (i,j) \in I \times J, \text{where } t_{ij} > T_k \\ &= 0, \quad (i,j) \in I \times J, \text{where } t_{ij} \leq T_k \\ c_{m+i,j} &= M, \quad (i,j) \in I \times J, \text{where } t_{m+i,j} > T^l \\ &= \lambda_l, \quad (i,j) \in I \times J, \text{where } t_{m+i,j} = T^l \\ &= \lambda_{l+1}, \quad (i,j) \in I \times J, \text{where } t_{m+i,j} = T^{l+1} \\ &\quad \vdots \qquad \vdots \\ &= \lambda_s, \quad (i,j) \in I \times J, \text{where } t_{m+i,j} = T^s \end{aligned}$$

As an M-feasible LOS of the problem $P_{L\beta}(T^{l-1})$ is a feasible solution of $CP_{L\beta}(T^k, T^l)$ and as $\sum_{j\in J} b_j > \sum_{i\in I} a_i$, it follows that optimal value in $CP_{L\beta}(T^k, T^l)$ will be non-zero. Corresponding to Stage-I shipment time T^k the optimal feasible solution of $CP_{L\beta}(T^k, T^l)$ provides the least Stage-II shipment time which is less than or equal to T^l. Thus, OBFS of $CP_{L\beta}(T^k, T^l)$ provides a feasible solution of the problem (P).

It may be observed that problems $P_{L\beta}(T^{l-1})$ and $CP_{L\beta}(T^k, T^l)$ for various values of l help in generating the pairs of the type $\left((T_1(.), T_2(.)) : T_1(.) > T_2(.)\right)$ for the Stage-I and Stage-II shipment times.

Similarly to generate the pairs of the type $\left((T_1(.), T_2(.)) : T_1(.) < T_2(.)\right)$, TMTPs of the type $P_{U\beta}(T^j)$ and CMTPs of the type $CP_{U\beta}(T^k, T^l)$ are studied, where $P_{U\beta}(T^j)$ is the TMTP derived from the problem P_β by abandoning the routes $(i,j) \in I \times J$ for which $t_{ij} \geq T^j$, $(i,j) \in I \times J$ and $CP_{U\beta}(T^k, T^l)$ is defined as:

$$\min_{\hat{S}} \sum_{I \times J} c_{ij} x_{ij} \qquad \ldots CP_{U\beta}(T^k, T^l)$$

where,

$$\begin{aligned} c_{m+i,j} &= M, \ (i,j) \in I \times J, \text{where } t_{m+i,j} > T^l \\ &= 0, \quad (i,j) \in I \times J, \text{where } t_{m+i,j} \leq T^l \\ c_{ij} &= M, \ (i,j) \in I \times J, \text{where } t_{ij} > T^k \\ &= \lambda_k, \ (i,j) \in I \times J, \text{where } t_{ij} = T^k \\ &= \lambda_{k+1}, (i,j) \in I \times J, \text{where } t_{ij} = T^{k+1} \\ &\ \ \vdots \qquad\qquad \vdots \\ &= \lambda_s, \ (i,j) \in I \times J, \text{where } t_{ij} = T^s \end{aligned}$$

Procedure for Two Stage Interval TMTP

A LOS, say X, of the problem P_β is obtained. Let $T(X) = \max_{I \times J}\left(\hat{t}_{ij}(x_{ij})\right) = T^r$. This T^r corresponds to either Stage-I shipment time or Stage-II shipment time. Without loss of generality let it correspond to Stage-I shipment time. Let Stage-II shipment time corresponding to this LOS be $T^{r+q_0^0} \leq T^r$ where q_0^0 is non-negative integer. To find the minimum Stage-II shipment time corresponding to the Stage-I shipment time T^r, solve the cost minimizing transportation problem $CP_{L\beta}(T^{r-p_o}, T^{r+q_0^0})$, where $p_0 = 0$. Its OBFS will yield the minimum Stage-II shipment time, say T^{r+q_0}, corresponding to time T^{r-p_0} of Stage-I shipment, where $q_0 (\geq q_0^0)$ is a non-negative integer. The first recorded pair thus obtained is (T^{r-p_0}, T^{r+q_0}) yielding value $T^{r-p_0} + T^{r+q_0}$ of the objective function of the Two Stage Interval TMTP (P). Suppose the pairs thus recorded so far for Stage-I and Stage-II shipment times be $\left((T^{r-p_j}, T^{r+q_j}) : T^{r-p_j} > T^{r+q_j}\right)$ $j = 0,1,\ldots,k$. For further generation of such pairs solve the restricted version $P_{L\beta}(T^{r+q_k})$ of the problem (P_β) wherein the routes $(m+i, j), (i,j) \in I \times J$ for which $t_{ij} \geq T^{r+q_k}$ are abandoned. If optimal feasible solution of $P_{L\beta}(T^{r+q_k})$ is an MFS, then the corresponding Stage-I shipment time, say $T^{r-p_{k+1}}$, will be more than T^{r-p_k} and Stage-II shipment time, say $T^{r+q_{k+1}^0}$, will be smaller than T^{r+q_k}. To find the minimum Stage-II shipment time corresponding to the Stage-I shipment time $T^{r-p_{k+1}}$, solve $CP_{L\beta}(T^{r-p_{k+1}}, T^{r+q_{k+1}^0})$. Its OBFS will yield the pair $(T^{r-p_{k+1}}, T^{r+q_{k+1}}), q_{k+1} \geq q_{k+1}^0$ for Stage-I and Stage-II shipment times. It is claimed that in these recorded pairs Stage-I shipment time is also the minimum corresponding to Stage-II shipment time. If, however, LOS of $P_{L\beta}(T^{r+q_k})$ is not an M-feasible solution, then it follows that Stage-II shipment time cannot be further reduced below T^{r+q_k} and the current best value of the objective function of the Two Stage Interval TMTP (P) is $\min_{j=0,1,\ldots,k}[T^{r-p_j} + T^{r+q_j}]$. It will be established that if the LOS, say $\underline{X}^{r+q_{k+1}^0}$, of the problem $P_{L\beta}(T^{r+q_k})$ is not an M-feasible solution, then

$$T_1(\underline{X}^{r+q_{k+1}^0})+T_2(\underline{X}^{r+q_{k+1}^0}) \geq \min_{j=0,1,\ldots k}[T^{r-p_j}+T^{r+q_j}]$$

which in turn would mean that there can not exist any other pair $\big((T_1(.),T_2(.)):T_1(.)\geq T_2(.)\big)$ yielding value of the sum of Stage-I and Stage-II shipment times less than $\min_{j=0,1,\ldots,k}[T^{r-p_j}+T^{r+q_j}]$. It may be observed that for these recorded pairs $T^{r-p_j} > T^{r-p_{j-1}}$ and $T^{r+q_j} < T^{r+q_{j-1}}$, $j=1,2,\ldots,k$.

Next, if possible, the pairs $\big((T_1(.),T_2(.)):T_1(.)<T_2(.)\big)$ in which Stage-I shipment time is less than the Stage-II shipment time are generated.

First, the restricted version, call it $P_{U\beta}(T^{r+q_0})$, of the problem (P_β) is constructed by abandoning the routes $(i,j)\in I\times J$ for which $t_{ij}\geq T^{r+q_0}$ i.e., setting $t_{ij}=M$, $(i,j)\in I\times J$ for which $t_{ij}\geq T^{r+q_0}$. If LOS of $P_{U\beta}(T^{r+q_0})$ is not an M-feasible solution, then it is claimed that there does not exist a pair $\big((T_1(.),T_2(.)):T_1(.)<T_2(.)\big)$ such that the corresponding value of the objective function of the Two Stage Interval TMTP (P) is better than $\min_{j=0,1,\ldots,k}(T^{r-p_j}+T^{r+q_j})$.

On the other hand, if LOS, say $\overline{X}^{r+\tilde{q}_0^0}$, of the problem $P_{U\beta}(T^{r+q_0})$ is an M-feasible solution, then the corresponding Stage-I shipment time, say $T^{r+\tilde{q}_0^0}$, would be less than T^{r+q_0}. Let Stage-II shipment time at this M-feasible LOS of the problem $P_{U\beta}(T^{r+q_0})$ be $T^{r-\tilde{p}_0}$. To obtain the minimum Stage-I shipment time corresponding to the time $T^{r-\tilde{p}_0}$ of Stage-II shipment, the cost minimizing transportation problem $CP_{U\beta}(T^{r+\tilde{q}_0^0},T^{r-\tilde{p}_0})$ is solved. Its OBFS yields the minimum Stage-I shipment time, say $T^{r+\tilde{q}_0}$, corresponding to the Stage-II shipment time $T^{r-\tilde{p}_0}$. Suppose the pairs $\big((T^{r+\tilde{q}_j},T^{r-\tilde{p}_j}): T^{r+\tilde{q}_j} < T^{r-\tilde{p}_j}\big)$, $j=0,1,\ldots,k$ have been generated so far. Existence of next such pair is examined by solving the restricted version $P_{U\beta}(T^{r+\tilde{q}_k})$ derived from P_β by setting $t_{ij}=M(>>0),(i,j)\in I\times J$ for which $t_{ij}\geq T^{r+\tilde{q}_k}$. If LOS, say $\overline{X}^{r+\tilde{q}_{k+1}^0}$, of $P_{U\beta}(T^{r+\tilde{q}_k})$ is an M-feasible solution, then note $T_1(\overline{X}^{r+\tilde{q}_{k+1}^0})\equiv T^{r+\tilde{q}_{k+1}^0}$ and $T_2(\overline{X}^{r+\tilde{q}_{k+1}^0})\equiv T^{r-\tilde{p}_{k+1}}$. To find the minimum Stage-I shipment time corresponding to the Stage-II shipment time $T^{r-\tilde{p}_{k+1}}$, the cost minimizing transportation problem $CP_{U\beta}(T^{r+\tilde{q}_{k+1}^0},T^{r-\tilde{p}_{k+1}})$ is solved. Its OBFS yields the minimum Stage-I shipment time (call it $T^{r+\tilde{q}_{k+1}}$) corresponding to the Stage-II shipment time, $T^{r-\tilde{p}_{k+1}}$. Thus, OBFS of $CP_{U\beta}(T^{r+\tilde{q}_{k+1}^0},T^{r-\tilde{p}_{k+1}})$ provides the pair $(T^{r+\tilde{q}_{k+1}},T^{r-\tilde{p}_{k+1}})$. On the other hand, if the LOS $\overline{X}^{r+\tilde{q}_{k+1}^0}$ of $P_{U\beta}(T^{r+\tilde{q}_k})$ is not an MFS, then Stage-I shipment time can not be reduced below $T^{r+\tilde{q}_k}$ and it is established that

$$T_1(\overline{X}^{r+\tilde{q}_{k+1}^0})+T_2(\overline{X}^{r+\tilde{q}_{k+1}^0}) \geq \min_{j=0,1,\ldots,k}[T^{r+\tilde{q}_j}+T^{r-\tilde{p}_j}]$$

which in turn means that no more pairs $\big((T_1(.),T_2(.)):T_1(.)<T_2(.)\big)$ can be obtained yielding value of the sum of Stage-I and Stage-II shipment times less than $\min_{j=0,1,\ldots,k}[T^{r+\tilde{q}_j}+T^{r-\tilde{p}_j}]$. It is claimed that in the pairs $\big((T^{r+\tilde{q}_j},T^{r-\tilde{p}_j}):T^{r+\tilde{q}_j}<T^{r-\tilde{p}_j}\big)$, $j=0,1,\ldots,k$ thus generated, Stage-II shipment time is also the minimum corresponding to Stage-I shipment time. Hence these pairs also correspond to feasible solutions of the Two Stage Interval Time Minimizing Transportation Problem.

Thus, the global minimum value of the objective function of the Two Stage Interval TMTP is

$$\min\left\{\min_{j\geq 0}\left(T^{r-p_j}+T^{r+q_j}\right),\min_{j\geq 0}\left(T^{r+\tilde{q}_j}+T^{r-\tilde{p}_j}\right)\right\}$$

To give the above stated procedure a sound mathematical foundation, the various claims are established in the following theorems.

Theorem 3 In a pair (T^{r-p_k}, T^{r+q_k}) corresponding to the OBFS of the problem $CP_{L\beta}(T^{r-p_k}, T^{r+q_k^0})$, T^{r-p_k} is the minimum Stage-I shipment time corresponding to time T^{r+q_k} of Stage-II shipment, where T^{r-p_k} and $T^{r+q_k^0}$ are the Stage-I and Stage-II shipment times respectively corresponding to the M-feasible LOS of $P_{L\beta}(T^{r+q_{k-1}})$.

Proof To prove that the Stage-I shipment time T^{r-p_k} is the minimum corresponding to the Stage-II shipment time T^{r+q_k}, assume the contrary. Suppose that T^{r-p_k} is not the minimum shipment time for Stage-I corresponding to time T^{r+q_k} of Stage-II shipment.

This implies that there exists a solution, say $\hat{X}$, of the problem (P_β) such that $T_1(\hat{X}) = T^{r-\hat{p}} < T^{r-p_k}$ and $T_2(\hat{X}) = T^{r+q_k}$

Clearly $T^r \le T_1(\hat{X}) < T^{r-p_k}$ as $T^r (= T^{r-p_0})$ is the minimum shipment time for Stage-I yielded by the LOS of the problem (P_β). M-feasible LOS of $P_{L\beta}(T^{r+q_{k-1}})$ yields the Stage-I shipment time T^{r-p_k}, which is also the overall shipment time for this TMTP. Also by definition of $P_{L\beta}(T^{r+q_{k-1}})$ it follows that $\hat{X}$ is its feasible solution.

By assumption $\hat{X}$ yields Stage-I shipment time $T^{r-\hat{p}} (< T^{r-p_k})$ and Stage-II shipment time T^{r+q_k}. Thus $\hat{X}$ yields overall shipment time for the time minimizing transportation problem $P_{L\beta}(T^{r+q_{k-1}})$ smaller than the one yielded by its LOS, which cannot be true. Hence T^{r-p_k} is the minimum Stage-I shipment time corresponding to the Stage-II shipment time T^{r+q_k}. ■

Theorem 4 In a pair $(T^{r+\tilde{q}_k}, T^{r-\tilde{p}_k})$ corresponding to the OBFS of the problem $CP_{U\beta}(T^{r+\tilde{q}_k^0}, T^{r-\tilde{p}_k})$, $T^{r-\tilde{p}_k}$ is the minimum Stage-II shipment time corresponding to time $T^{r+\tilde{q}_k}$ of Stage-I shipment, where $T^{r+\tilde{q}_k^0}$ and $T^{r-\tilde{p}_k}$ are the Stage-I and Stage-II shipment times corresponding to the M-feasible LOS of $P_{U\beta}(T^{r+\tilde{q}_{k-1}})$.

Proof The proof is similar to the proof of the theorem 3. ■

Theorem 5 If LOS, say $\underline{X}^{r+q_{k+1}^0}$, of the time minimizing transportation problem $P_{L\beta}(T^{r+q_k})$ is not an MFS, then

$$T_1(\underline{X}^{r+q_{k+1}^0}) + T_2(\underline{X}^{r+q_{k+1}^0}) \ge \min_{j=0,1,\dots,k} \{T^{r-p_j} + T^{r+q_j}\}$$

where, each of problems $P_{L\beta}(T^{r+q_j})$ $\forall\, j = 1,2,\dots,k-1$ has M-feasible LOS.

Proof As LOS of $P_{L\beta}(T^{r+q_k})$ is not an M-feasible solution; the Stage-II shipment time can not be further reduced below T^{r+q_k}. Currently best value of the sum of the shipment times in Stage-I and Stage-II is $\min_{j=0,1,\dots,k}\left[T^{r-p_j} + T^{r+q_j}\right]$. At non M-feasible LOS of $P_{L\beta}(T^{r+q_k})$ either (i) the Stage-I shipment time is one of the first $(k+1)$ recorded times: $T^{r-p_0}, T^{r-p_1}, T^{r-p_2}, \dots, T^{r-p_k}$, in which case the corresponding minimum Stage-II shipment time is already known or (ii) the Stage-I shipment time is none of the first $(k+1)$ recorded times but it lies in the interval $[T^{r-p_0}, T^{r-p_k}]$, in which case one of the recorded Stage-I and Stage-II shipment times would yield a smaller value of the sum of the shipment times or (iii) the Stage-I shipment time is more than T^{r-p_k}, in which case $\min_{j=0,1,\dots,k}\left[T^{r-p_j} + T^{r+q_j}\right]$ will be smaller than the sum of the Stage-I and Stage-II shipment times at the current non M-feasible LOS of $P_{L\beta}(T^{r+q_k})$. Hence the sum of the Stage-I and Stage-II shipment times corresponding to a non M-feasible LOS of the problem $P_{L\beta}(T^{r+q_k})$ is not less than the current best value of the sum. That is,

$$T_1(\underline{X}^{r+q_{k+1}^0}) + T_2(\underline{X}^{r+q_{k+1}^0}) \ge \min_{j=0,1,\dots,k}\left[T^{r-p_j} + T^{r+q_j}\right]$$

■

Remark If LOS of the problem $P_{L\beta}(T^{r+q_j})$ is not an MFS, then no further restricted version of the problem (P_β), namely $P_{L\beta}(T^{r+q_j})$ $j \geq k$ can provide a solution of the problem (P) yielding value better than $\min_{j=0,1,\ldots,k}\left(T^{r-p_j} + T^{r+q_j}\right)$.

Remark Let pairs in hand of Stage-I and Stage-II shipment times be (T^{r-p_j}, T^{r+q_j}), $j = 0,1,\ldots,k$. Let LOS of the problem $P_{L\beta}(T^{r+q_k})$ be an MFS. Then, Stage-I shipment time corresponding to this M-feasible LOS is more than T^{r-p_k} since for a given Stage-I shipment time less than or equal to T^{r-p_k} the Stage-II shipment time can not be less than T^{r+q_k}. (Recall that M-feasible solution LOS of $P_{L\beta}(T^{r+q_k})$ yields Stage-II shipment time less than T^{r+q_k}.)

Theorem 6 If LOS of the problem $P_{U\beta}(T^{r+q_0})$, say $\overline{X}^{r+\tilde{q}_0^0}$, is not an MFS, then $T_1(\overline{X}^{r+\tilde{q}_0^0}) + T_2(\overline{X}^{r+\tilde{q}_0^0}) \geq \min_{j\geq 0}\{T^{r-p_j} + T^{r+q_j}\}$

.*Proof* As $\overline{X}^{r+\tilde{q}_0^0}$ is not an MFS, we have $\overline{x}_{ij}^{r+\tilde{q}_0^0} > 0$ for some $(i,j) \in I \times J$ for which $t_{ij} \geq T^{r+q_0}$, i.e., $T_1(\overline{X}^{r+\tilde{q}_0^0}) \geq T^{r+q_0}$.

Also we have $T_2(\overline{X}^{r+\tilde{q}_0^0}) \geq T^r = T^{r-p_0}$ (since $p_0 = 0$). Therefore,

$$T_1(\overline{X}^{r+\tilde{q}_0^0}) + T_2(\overline{X}^{r+\tilde{q}_0^0}) \geq T^{r+q_0} + T^{r-p_0}$$
$$\geq \min_{j\geq 0}\{T^{r-p_j} + T^{r+q_j}\}$$

Hence the result. ■

Remark If LOS of the problem $P_{U\beta}(T^{r+q_0})$ is not an MFS, then the restricted versions $P_{U\beta}(T^{r+\tilde{q}_j})$ $\forall\, j \geq 0$ of the problem P_β can not provide an optimal solution of the problem (P). This also implies that there does not exist a feasible solution of the problem (P) having Stage-I time less than T^{r+q_0}.

Theorem 7 If LOS, say $\overline{X}^{r+\tilde{q}_0^{k+1}}$, of $P_{U\beta}(T^{r+\tilde{q}_k})$ is not an MFS, then $T_1(\overline{X}^{r+\tilde{q}_{k+1}^0}) + T_2(\overline{X}^{r+\tilde{q}_{k+1}^0}) \geq \min_{j=0,1,\ldots,k}\{T^{r+\tilde{q}_j} + T^{r-\tilde{p}_j}\}$ where, each of the problem $P_{U\beta}(T^{r+\tilde{q}_j})$ $\forall\, j = 1,2,\ldots k-1$ has an M-feasible LOS.

Proof The proof is similar to the proof of the theorem 5. ■

Remark If LOS of the problem $P_{U\beta}(T^{r+\tilde{q}_k})$ is not an MFS, then no further restricted version of (P_β), namely $P_{U\beta}(T^{r+\tilde{q}_j}), j \geq k$ can provide a solution of the problem (P) yielding value better than $\min_{j=0,1,\ldots,k}\left(T^{r+\tilde{q}_j} + T^{r-\tilde{p}_j}\right)$.

Remark If LOS of the problem $P_{U\beta}(T^{r+\tilde{q}_j})$ is an MFS for all $j = 1,2,\ldots,k$, then as $T_1(\overline{X}^{r+\tilde{q}_{k+1}^0}) < T^{r+\tilde{q}_k}$, it follows that $T_2(\overline{X}^{r+\tilde{q}_{k+1}^0}) > T^{r-\tilde{p}_k}$ since if $T_2(\overline{X}^{r+\tilde{q}_{k+1}^0}) \leq T^{r-\tilde{p}_k}$ then corresponding minimum Stage-I time will be greater than or equal to $T^{r+\tilde{q}_k}$.

The next theorem proves that the proposed solution methodology indeed obtains the global optimal solution of the Two Stage Interval TMTP (P).

Theorem 8 If the generated pairs of Stage-I and Stage-II shipment times are $\left(T^{r-p_k}, T^{r+q_k}\right), k \geq 0$ and $\left(T^{r+\tilde{q}_k}, T^{r-\tilde{p}_k}\right)$, $k \geq 0$ then, the optimal value of the objective function of the problem (P) is

$$\min\left\{\min_{k\geq 0}\left(T^{r-p_k} + T^{r+q_k}\right), \min_{k\geq 0}\left(T^{r+\tilde{q}_k} + T^{r-\tilde{p}_k}\right)\right\}$$

Proof If the theorem is not to be true, then there must exist a feasible solution, say X_G, of the problem (P) such that the corresponding Stage-I and Stage-II shipment times (call them $T_1(X_G)$ and $T_2(X_G)$ respectively) are such that

$$T_1(X_G)+T_2(X_G)<\min\left\{\begin{array}{l}\min\limits_{k\geq 0}\left(T^{r-p_k}+T^{r+q_k}\right),\\ \min\limits_{k\geq 0}\left(T^{r+\tilde{q}_k}+T^{r-\tilde{p}_k}\right)\end{array}\right\}$$

As X_G is a feasible solution of the problem (P), it follows that $T_2(X_G)$ is the minimum Stage-II shipment time corresponding to the Stage-I shipment time $T_1(X_G)$. Therefore, X_G is an M-feasible solution of the problem (P_β).

The above inequality implies that $T_1(X_G)+T_2(X_G)<T^{r-p_0}+T^{r+q_0}$.
Without loss of generality, assume that $T_1(X_G)\geq T_2(X_G)$. As $T^{r-p_0}(=T^r)$ is the optimal transportation time for the time minimizing transportation problem (P_β), it follows that $T_1(X_G)\geq T^{r-p_0}$ and hence $T_2(X_G)<T^{r+q_0}$. Therefore, there exists an index, say d (a positive integer not less than 1), such that
$T^{r+q_d}<T_2(X_G)<T^{r+q_{d-1}}<T^{r+q_0}$
This implies that X_G is an M-feasible solution of the problem $P_{L\beta}(T^{r+q_{d-1}})$.
M-feasible LOS of $P_{L\beta}(T^{r+q_{d-1}})$ yields Stage-I shipment time T^{r-p_d}. By hypothesis,
$T_1(X_G)+T_2(X_G)<T^{r-p_d}+T^{r+q_d}$.
As $T_2(X_G)>T^{r+q_d}$, we have $T_1(X_G)<T^{r-p_d}$.
This implies that X_G is a solution better than the M-feasible LOS of $P_{L\beta}(T^{r+q_{d-1}})$, which is not true.

Hence there does not exist any feasible solution of (P) yielding sum of Stage-I and Stage-II shipment times less than

$$\min\left\{\min_{k\geq 0}\left(T^{r-p_k}+T^{r+q_k}\right),\min_{k\geq 0}\left(T^{r+\tilde{q}_k}+T^{r-\tilde{p}_k}\right)\right\}$$

■

The formal algorithm for the Two Stage Interval TMTP is given below.

Algorithm

Step 1 Obtain an LOS of the problem (P_β). Note the corresponding Stage-I time as $T^r=T^{r-p_0}$ and Stage-II time as $T^{r+q_0^0}$.
Solve cost minimizing transportation problem $CP_{L\beta}(T^{r-p_o},T^{r+q_0^0})$ to find the minimum Stage-II shipment time, say T^{r+q^0}, of Stage-II corresponding to the time T^{r-p_0} of Stage-I shipment. Record this pair as (T^{r-p_0},T^{r+q_0}).
If $T^{r-p_0}=T^1$ or $T^{r+q_0}=T^s$, then stop and go to step 3. Else, go to step 2.

Step 2 $(k\geq 1)$ Construct the problem $P_{L\beta}(T^{r+q_{k-1}})$ and find its LOS. If it is not an MFS, then go to step 3. Else, solve the cost minimizing transportation problem $CP_{L\beta}(T^{r-p_k},T^{r+q_k^0})$ to find the minimum Stage-II shipment time T^{r+q_k} corresponding to the time T^{r-p_k} of Stage-I shipment.
Record the pair (T^{r-p_k},T^{r+q_k}).
If $T^{r-p_k}=T^1$ or $T^{r+q_k}=T^s$, then stop and go to step 3. Else, execute step 2 for next higher value of k.

Step 3 Construct the problem $P_{U\beta}(T^{r+q_0})$ and obtain its LOS. If it is not an M-feasible solution, then go to step 5. Else, note the Stage-I shipment time as $T^{r+\tilde{q}_0^0}$ and Stage-II time as $T^{r-\tilde{p}_0}$. To find the minimum Stage-I shipment time, $T^{r+\tilde{q}_0}$, corresponding to time $T^{r-\tilde{p}_0}$ of Stage-II shipment, solve the cost minimizing transportation problem $CP_{U\beta}(T^{r+\tilde{q}_0^0},T^{r-\tilde{p}_0})$.
Record the pair $(T^{r+\tilde{q}_0},T^{r-\tilde{p}_0})$.
If $T^{r+\tilde{q}_0}=T^s$ or $T^{r-\tilde{p}_0}=T^1$, then stop and go to step 5. Else, go to step 4.

Step 4 $(k\geq 1)$ Construct the problem $P_{U\beta}(T^{r+\tilde{q}_{k-1}})$ and find its LOS. If it is not an M-feasible solution, then go to step 5. Else, note the Stage-I shipment time as $T^{r+\tilde{q}_k^0}$ and Stage-II shipment time as $T^{r-\tilde{p}_k}$.

Solve cost minimizing transportation problem $CP_{U\beta}(T^{r+\tilde{q}_k^0}, T^{r-\tilde{p}_k})$ to find the minimum Stage-I shipment time $T^{r+\tilde{q}_k}$ corresponding to Stage-II shipment time, $T^{r-\tilde{p}_k}$. Record the pair $(T^{r+\tilde{q}_k}, T^{r-\tilde{p}_k})$.

If $,T^{r-\tilde{p}_k} = T^1$ or $T^{r+\tilde{q}_k} = T^s$, then stop and go to step 5. Else, repeat this step for next higher value of k.

Step 5 Find

$$\min\left\{ \min_{k\geq 0}\left(T^{r-p_k} + T^{r+q_k}\right), \min_{k\geq 0}\left(T^{r+\tilde{q}_k} + T^{r-\tilde{p}_k}\right)\right\}.$$

This will be the optimal value of the objective function of the problem (P).

Numerical Illustration

Consider the 3×6 Two Stage Interval TMTP given below in Table 1.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	a'_i
S_1	26 $= t_{11}$	23	59	38	19	20	6	8
S_2	40	48	20	19	23	59	15	29
S_3	26	38	48	20	19	40	12	18
b_j	6	9	3	14	10	5		

Table 1

where S_i is the i^{th} source, $i = 1,2,3$ and D_j is the j^{th} destination, $j = 1,2,\ldots,6$.

The partition of transportation times on various routes is:

$$T^1(=59) > T^2(=48) > T^3(=40) > T^4(=38) >$$
$$T^5(=26) > T^6(=23) > T^7(=20) > T^8(=19)$$

$T^s = T^8 = 19$ and therefore, $s = 8$.

The corresponding (P_β) problem is given below in Table 2.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	a_i
S_1	26	23	59	38	19	20	M	6
S_2	40	48	20	19	23	59	M	15
S_3	26	38	48	20	19	40	M	12
S_4	26	23	59	38	19	20	0	2
S_5	40	48	20	19	23	59	0	14
S_6	26	38	48	20	19	40	0	6
b_j	6	9	3	14	10	5	8	

Table 2

LOS of the problem (P_β) yields the Stage-I shipment time as 26 and Stage-II shipment time as 38.

Therefore, $T^r = T^4 = 38$ and hence $r = 4$. To obtain the minimum Stage-I shipment time corresponding to Stage-II shipment time 38, solve the cost minimizing transportation problem $CP_{U\beta}(26,38)$. Its optimal solution yields the same pair (26,38). Hence the **first recorded pair of the Stage-I and Stage-II shipment times is (26,38)**.

To obtain a new pair, the time minimizing transportation problem $P_{U\beta}(26)$ is solved. Its LOS is M-feasible and yields the pair (23,40) of the Stage-I and Stage-II shipment times. The optimal solution of the cost minimizing transportation problem $CP_{U\beta}(23,40)$ gives back the pair (23,40).

Hence the **second recorded pair of the Stage-I and Stage-II shipment times is (23,40)**.

Next the time minimizing transportation problem $P_{U\beta}(23)$ is solved. Its LOS is not an M-feasible solution. Hence the time minimizing transportation problem $P_{L\beta}(26)$ is constructed whose LOS is M-feasible and yields the pair (38,23). Then, we solve the cost minimizing transportation problem $CP_{L\beta}(38,23)$ to obtain the minimum Stage-II shipment time corresponding to shipment time 38 of Stage-I. Optimal solution of $CP_{L\beta}(38,23)$ yields the Stage-II time as 20.

Hence the **third recorded pair of the Stage-I and Stage-II shipment times is (38,20)**.

Next, LOS of $P_{L\beta}(20)$ yields the pair (40,19). The cost minimizing transportation problem $CP_{L\beta}(40,19)$ gives back the pair (40,19). Since Stage-II time has reached $T^s = 19$ we stop here.
The **fourth recorded pair of the Stage-I and Stage-II shipment times is (40,19)**.

Now, min{26+38, 23+40, 38+20, 40+19}=58. Hence for the optimal solution of the problem (P) the Stage-I shipment time is 38 and Stage-II shipment time is 20 yielding the sum of Stage-I and Stage-II shipments as 58. Stage-I and Stage-II shipment schedules for this optimal solution are given in Table 3.

	D_1	D_2	D_3	D_4	D_5	D_6	D_7	a_i
S_1	26	(3) 23	59	38	19	(3) 20	M	6
S_2	(0) 40	48	20	(11) 19	(4) 23	59	M	15
S_3	(6) 26	(6) 38	48	20	19	40	M	12
S_4	26	23	59	38	19	(2) 20	0	2
S_5	40	48	(3) 20	(3) 19	23	59	(8) 0	14
S_6	26	38	48	20	(6) 19	40	0	6
b_j	6	9	3	14	10	5	8	

Table 3

Note that only 7 CMTPs $(< 4(s-r)-2 = 14)$ are to be solved.

Concluding Remarks

a. Two Stage Interval TMTP has been introduced for the first time and as such we are not aware of any solution strategy for the same and hence no comparative study could be carried out.

As the problem under study is a non-convex optimization problem, some sort of enumeration of feasible solutions has to be resorted to. To make enumeration a viable option very judicious enumeration is proposed.

b. In the developed algorithm not more than $(4(s-r)-2)$ number of CMTPs are solved to generate the different pairs of Stage-I and Stage-II shipment times.

It is known that a CMTP is solvable in polynomial running time $O(m \log n(m + n \log n))$ (Orlin, 1988). Hence the proposed algorithm is also a polynomial time algorithm.

c. The algorithm proposed for Two Stage Interval TMTP has been coded in C++ and verified successfully with the help of a lot of test problems of various sizes. Recordings of some of these examples are listed in Table 4.

Size of the problem	No of partitions	Longest Duration Time	Shortest Duration Time	No. of		No of pairs obtained	Optimal pairs(s)	Optimal value
				TMTPs solved	CMTPs solved			
2×4	9	29	1	3	1	1	(14,2)	16
2×8	15	30	4	4	2	2	(13,21)	34
3×5	9	10	1	4	2	2	(7,7)	14
3×9	12	10	1	4	2	2	(7,3)	14
4×5	14	21	1	4	3	1	(8,1)	9
5×5	18	21	2	4	2	2	(13,6), (15,4)	19
6×7	17	18	1	5	3	3	(11,3)	14
7×8	25	30	1	5	3	3	(14,6)	20
7×9	20	18	1	5	3	3	(7,2)	9
8×7	20	18	1	5	3	3	(9,6)	15
8×8	19	17	1	5	3	3	(6,6), (9,3)	12
8×10	20	18	1	5	3	3	(8,4)	12
9×7	19	18	1	6	4	4	(4,8)	12
9×9	19	18	1	4	2	2	(9,5)	14
10×9	14	12	1	4	2	2	(5,6)	11
10×10	20	18	1	4	2	2	(4,10)	14
10×11	20	18	1	4	2	2	(3,11),(12,2)	14
10×12	20	18	1	5	3	3	(6,2)	8
10×14	20	18	1	5	3	3	(4,5)	9
10×15	20	18	1	6	4	4	(6,5), (7,4), (4,7)	11
12×10	20	17	1	5	3	3	(9,6)	15
15×10	20	17	1	3	1	2	(6,2)	8
20×5	20	17	1	3	1	1	(12,7)	19

Table-4

Acknowledgement

We are indebted to Prof. S. N. Kabadi, University of New Brunswick-Fredericton, Canada, with whom we had very useful discussions.

We are also thankful to Mr. Avneet Bansal who greatly helped us in implementation of the proposed algorithm.

References

[1] R. K. Ahuja, Algorithms for the Minimax Transportation Problem, Naval Research Logistics Quarterly, vol. 33, pp 725-739, (1986).

[2] J. Amaya and G. De. Ghellinck, "Duality for Inexact Linear Programming Problems," *Optimization*, vol. 39, pp 137-150, 1997.

[3] S. Bansal and M. C. Puri, "A Min-max Problem," *ZOR*, vol. 24, pp 191-200, 1980.

[4] H. L. Bhatia, Kanti Swarup and M. C. Puri, "Time-Cost Trade-off in a Transportation Problem," *Opsearch*, vol. 13 (3-4), pp 129-142, 1976.

[5] A. Charnes and F. A. Granot, "Modified Algorithm for Solving Interval Linear Programming Problems," *Cahier*, vol. 18(3), pp 323-335, 1976.

[6] T. Gal, *Post Optimal Analysis, Parametric Programming and Related Topics*, McGraw-Hill, New York, 1979.

[7] R. S. Garfinkel and M. R. Rao, "Bottleneck Transportation Problem," *Naval Research Logistics Quarterly*, vol. 18, pp 465-472, 1971.

[8] P. L. Hammer, "Time Minimizing Transportation Problem," *Naval Research Logistics Quarterly*, vol. 16, pp 345-357, 1969.

[9] S. Khanna, H. C. Bakshi and M. C. Puri, "On Controlling Flow in Transportation Problems," *Scientific Management of Transport Systems*, edited by N.K.Jaiswal, North Holland, The Netherlands, pp 293-301, 1981.

[10] S. Khanna and M. C. Puri, "Solving a Transportation Problem with Mixed Constraints and Specified Transportation Flow," *Opsearch*, vol. 20(1), pp 16-24, 1983.

[11] J. B. Mazzola and A. W. Neebe, "An algorithm for the Bottleneck Generalized Assignment Problem", *Computers and Operations Research*, vol. 20(4), pp 355-362, 1993.

[12] J. B. Orlin, "A Faster Strongly Polynomial Minimum Cost Flow Algorithm," Proc. of the 20th ACM Symposium on the theory of computing, vol. 11, pp 377-387, 1988.

[13] P. D. Robers and A. Ben Israel, "A Suboptimization Method for Interval Linear Programming: A New Method for Linear Programming," *Linear Algebra and its Applications*, vol. 3, pp 383-405, 1970.

[14] Satya Prakash, "On Minimizing the Duration of Transportation," Proc. of Indian Academy of Science, vol. 91(1), pp 53-57, 1982.

[15] H. D. Sherali, "Equivalent Weights for Lexicographic Multi-Objective Programs: Characterization and Computations," *European Journal of Operational research*, vol. 11, pp 367-379, 1982.

[16] A. L. Soyster, "Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming," *Operations Research,* vol. 21, pp 1154-1157, 1973.