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COMBINATORIAL PROBLEM INVOLVING SUM OF A FINITE NUMBER OF BOTTLENECKS AS OBJECTIVE

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Minimization of the sum of a finite number of concave bottleneck functions subject to linear constraints is studied in the present paper. A related bottleneck linear programming problem which minimizes a single bottleneck objective is constructed whose extreme point solutions provide bounds on the optimal value of the objective function of the problem under consideration. Some k^{th} best solution $(k \geq 1)$ of this bottleneck linear programming problem satisfying certain conditions is shown to provide an optimal feasible solution of the problem. The proposed algorithm obtains the global optimal solution of the main problem in a finite number of steps. A constrained version of this problem where the optimal feasible solution is required to satisfy an additional constraint is also discussed.

Keywords: bottleneck programming, convex programming, extreme point ranking, min-max programming.

1. Introduction

Bottleneck linear programming problems deal with minimization of a concave bottleneck objective function, or with maximization of a convex bottleneck objective function over a closed convex region bounded by hyperplanes. Bansal et. al. (1980), Gupta et. al. (1982), Garfinkel et. al. (1971), Bhatia et. al. (1976, 1977), Hammer (1969) and many others have studied the bottleneck linear programming problem with a single concave bottleneck function as objective. The Time Minimization Transportation Problem, which is a special case of bottleneck linear programming problem, has been studied by Hammer (1969), Szwarc (1971), Garfinkel et. al. (1971) and Bhatia et. al. (1976, 1977). A method for finding the k^{th} $(k \geq 1)$ best extreme point solution and its alternates, for the bottleneck linear program-

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ming problem, has been developed by Bansal et. al. (1980).

The present paper studies minimisation of the sum of a finite number of concave bottleneck objective functions subject to linear constraints. Mathematically, the problem may be stated as:

$$\min_{X \in S} F(X), \tag{P1}$$

where

$$S = \{X \in \mathbb{R}^n / AX = b,\ X \geq 0\},\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m,$$

$$F(X) = \sum_{i=1}^{r} F_i(X),$$

and $F_i(X) = \max_{1 \leq j \leq n} f_j^i(x_j)$, $f_j^i(x_j)$ being defined as:

$$\begin{aligned} f_j^i(x_j) &= f_j^i, \quad \text{if } x_j > 0, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Such problems arise in situations where a number of products are being produced and they have to pass through r phases of production successively, such that each phase begins only when the previous one has ended. Let X be a feasible solution with respect to the constraints $AX = b$, $X \geq 0$. Then the components x_j, $j = 1, \ldots, n$ of X give the quantity of various goods to be produced, and $f_j^i(x_j)$ gives the time spent on the j^{th} product in the i^{th} phase, irrespective of the quantity x_j of the product produced. Then

$$F_i(X) = \max_j f_j^i(x_j)$$

is the time spent in the i^{th} phase with respect to X, and total production time is

$$F(X) = \sum_{i=1}^{r} F_i(X).$$

An optimal solution to the problem gives the least possible production time.

A transportation problem in which a complete schedule consists of an onward and a return journey, with the restriction that all the return trips on the various routes can start only when all the onward trips have ended, also lends itself to solution by the method developed here. In this case, the objective function becomes the sum of two bottleneck functions, the first giving the time taken to complete the onward journeys and the second giving the time taken to complete the return journeys. It is being assumed that all the carriers start simultaneously for the onward journeys and also for the return journeys. On any route, the time taken for the onward journey is taken to be different from the time taken for the return journey, to accommodate the possibility of different modes of transportation being used on the two journeys.

In the next section on theoretical development, a related bottleneck linear programming problem is constructured whose k^{th} (for some $k \geq 1$) best extreme point solution satisfying certain conditions is proved to yield an optimal feasible solution of (P1). In section 3, a finite algorithm is developed to obtain a global optimal solution of the stated problem by successively tightening the bounds on the value of the objective function. A constrained version of (P1) where an additional constraint is imposed on the system is presented in section 4. A numerical illustration is also included.

2. Theoretical Development

The functions $F_i(X)$, $i = 1, ..., r$ are concave (Bansal et. al. (1980); Mangasarian (1969)). Thus the objective function $F(X)$ of $(P1)$, being the sum of a finite number of concave functions, is concave and hence its global minimum is attained at an extreme point of S (Mangasarian (1969); Murty (1976)). This motivates the investigation of only the extreme points of S for finding an optimal solution of (P1).

A linear min-max problem related to (P1) is:

$$\underset{X \in S}{\text{Min}}\ R(X), \qquad (P2)$$

where $R(X) = \underset{j}{\text{Max}}\ r_j(x_j)$, $r_j(x_j)$ being defined for each $j = 1, ..., n$ as $r_j(x_j) = \sum_{i=1}^{r} f_j^i(x_j)$.

Theorem 1. $R(X) \leq F(X)\ \forall\ X \in S$.

Proof. For any $X = [x_j] \in S$,

$$\begin{aligned} r_j(x_j) &= \sum_{i=1}^{r} f_j^i(x_j) \\ &\leq \sum_{i=1}^{r} \left(\underset{1 \leq j \leq n}{\text{Max}} f_j^i(x_j) \right) \\ &= \sum_{i=1}^{r} F_i(X) \\ &= F(X). \end{aligned}$$

Therefore, $\underset{j}{\text{Max}}\ r_j(x_j) \leq F(X)\ \forall\ X \in S$. Hence $R(X) \leq F(X)\ \forall\ X \in S$.

Notations.

$S_k = \{X_{ku}\},\ u = 1, ..., n_k$, the set of kth best extreme point solutions of (P2).

$R^k = R(X),\ X \in S_k$.

$$F^k = \text{Min}\ \{F(X_{ju})/u = 1, ..., n_j,\ j = 1, ..., k\}$$
$$= \text{Min}\ \{F(X)/X \in \bigcup_{j=1}^{k} S_j\}.$$

Clearly, $F^1 = \underset{x \in S_1}{\text{Min}}\ F(X)$, and $F^k = \text{Min}\ [F^{k-1},\ \underset{x \in S_k}{\text{Min}}\ F(X)],\ k \geq 2$.

Theorem 2. If $R^1 = F^1$, then F^1 is the optimal vaue of $F(X)$.

Proof. As R^1 is the optimal value of $R(X)$,

$$R^1 \leq R(X)\ \forall\ X \in S.$$

Also, $R(X) \leq F(X)\ \forall\ X \in S$ (theorem 1)
Therefore,

$$F^1 = R^1$$
$$\leq F(X)\ \forall\ X \in S,$$

which implies that F^1 is the optimal value of $F(X)$. And if, $F^1 = \underset{x \in S_1}{\text{Min}}\ F(X) = F(X_{1,u_1})$ (say), then X_{1,u_1} is an optimal feasible solution of $(P1)$.

Theorem 3. For all $k \geq 1$, if $R^k < F^k \leq R^{k+1}$, then F^k is the optimal value of $F(X)$.

Proof. By definition of F^k,

$$F^1 \geq F^2 \geq ... \geq F^k. \tag{1}$$

Now,

$$F^k \leq R^{k+1} \qquad \text{(by hypothesis)},$$
$$\leq F(X)\ \forall\ X \in S_{k+1} \qquad \text{(theorem 1)},$$

which implies

$$F^k \leq \underset{x \in S_{k+1}}{\text{Min}}\ F(X).$$

Therefore,

$$F^{k+1} = \text{Min}\ \{F(X) : X \in \bigcup_{j=1}^{k+1} S_j\}$$
$$= F^k. \tag{2}$$

Also,

$$\begin{aligned} F^k &\leq R^{k+1} && \text{(by hypothesis)} \\ &< R^q \ \forall\, q \geq k+2 \\ &\leq F(X)\ \forall\, X \in S_q && \text{(theorem 1).} \end{aligned}$$

Therefore,

$$F^k < F(X)\ \forall\, X \in S_q \quad \text{and} \quad q \geq k+2. \tag{3}$$

It follows from (1), (2) and (3) that F^k is the optimal value of $F(X)$. And if $F^k = \text{Min}\ [F(X) : X \in \bigcup_{j=1}^{k} S_j] = F(\hat{X})$, then $\hat{X}$ is an optimal solution of (P1).

3. Algorithm

Step 1. Find S_1 (Bansal et. al. (1980); Garfinkel et. al. (1971); Hammer (1969)). If $R^1 = F^1$, then by theorem 2, F^1 is the optimal value of (P1).

If $R^1 < F^1$, then lower and upper bounds on the optimal value of (P1) are R^1 and F^1 respectively. Set $k = 2$ and go to step 2.

Step 2. Find S_k (Bansal et. al. (1980); Murty (1976)). If $F^{k-1} \leq R^k$, stop. By theorem 3, F^{k-1} is the optimal value of $F(X)$.

If $R^k < F^{k-1}$, find F^k and go to step 2 with next higher value of k.

Remark. As the algorithm ranks extreme points of S in non-descending order of values of $R(X)$ till an optimal extreme point solution of (P1) is reached, and as the extreme points of S are finite in number, the algorithm obtains the optimal solution of (P1) in a finite number of steps. However, it is not a polynomial-time algorithm, as an exponential number of extreme points of S may have to be ranked before the optimal solution is reached (Klee and Minty (1972)).

4. Constrained Version of Problem (P1)

If a budgetary or some other constraint is imposed on the system and it is not satisfied by any of the existing optimal feasible solutions obtained by the preceding algorithm, then the extreme point solutions of (P1) have to be ranked in increasing order of values of $F(X)$ till the one satisfying the additional constraint is obtained. By further ranking the extreme point solutions of (P2) in non-decreasing order of values of $R(X)$, the 2nd best, 3rd best,..., extreme point solutions of (P1) can be obtained. This proposed approach to tackle the constrained version of (P1) is validated by the results established as follows:

The constrained version of (P1) is

$$\operatorname*{Min}_{x\in s} F(X), \tag{$P1'$}$$

where X satisfies $g(X) \leq 0$, g being a numerical function defined on $\mathbb{R}^n$.

Theorem 4. The optimal feasible solution of $(P1')$ is attainable at an extreme point of S.

Proof. Clearly, if $X \in Int(S)$, the set of interior points of S, such that $g(X) = 0$, then there exists an $\hat{X} \notin Int(S)$ such that $g(\hat{X}) = 0$ and $F(\hat{X}) \leq F(X)$.

Let, if possible, an optimal feasible solution of $(P1')$ be a non-extreme point X_0 of S. This implies that X_0 lies on a facet, say $\bar{S}$, of S. Since any point (other than an extreme point) on a facet can be expressed as a convex combination of two points on that facet, one of which is an extreme point of S (Murty (1976)), it follows that there exists an extreme point $\hat{X}_0$ of S satisfying $g(X) \leq 0$ such that $F(\hat{X}_0) \leq F(X_0)$. Hence there exists an extreme point of S at which the optimal value of $F(X)$ in $(P1')$ can be attained.

Theorem 5. If for $r \geq 1$,

$$R^p < \mathrm{Min}[F(X) : X \in \bigcup_{j=1}^{p} S_j,\ F(X) > \bar{F}_r] \leq R^{p+1},\ p \geq 1,$$

then Min $[F(X) : X \in \bigcup_{j=1}^{p} S_j,\ F(X) > \bar{F}_r] = \bar{F}_{r+1}$, where $\bar{F}_r$ stands for the r^{th} best value of $F(X)$.

Proof.

$$\begin{aligned}\mathrm{Min}\ [F(X) : X \in \bigcup_{j=1}^{p} S_j,\ F(X) > \bar{F}_r] &\leq R^{p+1}\ (p, r \geq 1)\ \text{(by hypothesis)}\\ &< R^q\ \forall\ q \geq p+2\\ &\leq F(X)\ \forall\ X \in S_q\ \text{(theorem 1)},\end{aligned}$$

which implies

$$\mathrm{Min}\ [F(X) : X \in \bigcup_{j=1}^{p} S_j, F(X) > \bar{F}_r] < F(X)\ \forall\ X \in S_q,\quad q \geq p+2. \tag{4}$$

Again by hypothesis,

$$\text{Min } [F(X) : X \in \bigcup_{j=1}^{p} S_j, F(X) > \bar{F}_r] \leq R^{p+1}$$
$$\leq F(X) \; \forall \; X \in S_{p+1} \quad \text{(theorem 1)}.$$

Therefore, Min $[F(X) : X \in \bigcup_{j=1}^{p} S_j, \; F(X) > \bar{F}_r] \leq \underset{x \in s_{p+1}}{\text{Min}} \; F(X)$ which implies

$$\text{Min } [F(X) : X \in \bigcup_{j=1}^{p+1} S_j, F(X) > \bar{F}_r]$$
$$= \text{Min } [F(X) : X \in \bigcup_{j=1}^{p} S_j, F(X) > \bar{F}_r]. \tag{5}$$

(4) and (5) imply

$$\text{Min } [F(X) : X \in \bigcup_{j=1}^{p} S_j, F(X) > \bar{F}_r] = \bar{F}_{r+1}.$$

Remark. Let $R^1, R^2, ..., R^N$ be the distinct values of $R(X)$ over S such that $R^1 < R^2 < ... < R^N$.

(i) If $R^N <$ Min $[F(X) : X \in \bigcup_{j=1}^{N-1} S_j, F(X) > \bar{F}_r]$, then

$$\bar{F}_{r+i} = \text{Min } [F(X) : X \in \bigcup_{j=1}^{N} S_j, F(X) > \bar{F}_{r+i-1}], \quad i \geq 1.$$

(ii) If $R^n \geq$ Min $[F(X) : X \in \bigcup_{j=1}^{N-1} S_j, F(X) > \bar{F}_r]$, then $\bar{F}_{r+1} =$ Min $[F(X)$ $X \in \bigcup_{j=1}^{N-1} S_j, F(X) > \bar{F}_r]$ and $\bar{F}_{r+i} =$ Min $[F(X) : X \in \bigcup_{j=1}^{N} S_j, F(X) > \bar{F}_{r+i-1}], \quad i \geq 2$.

Example. Consider the problem

$$\begin{aligned} \text{Min} \quad & F(X) && (P1') \\ \text{subject to} \quad & 2x_1 + x_2 + 2x_3 + 3x_5 + x_6 = 4, \\ & x_1 + 2x_2 + x_4 + 2x_5 + 3x_6 = 6, \\ & x_1 + x_2 + x_4 + x_5 + 2x_6 = 4, \\ & x_j \geq 0, \; j = 1, ..., 6, \end{aligned}$$

where X satisfies $3x_1^2 + 4x_2^2 + 3x_3 + x_4 + 5x_5 + x_6 \leq 7$, and

$$f_1^1 = 10, \quad f_2^1 = 3, \quad f_3^1 = 21, \quad f_4^1 = 29, \quad f_5^1 = 15, \quad f_6^1 = 7$$
$$f_1^2 = 11, \quad f_2^2 = 8, \quad f_3^2 = 18, \quad f_4^2 = 6, \quad f_5^2 = 14, \quad f_6^2 = 20$$

The related min-max problem is

$$\text{Min } R(X) \tag{$P2$}$$

with the r_j' as

$$r_1 = 21, \quad r_2 = 11, \quad r_3 = 39, \quad r_4 = 35, \quad r_5 = 29, \quad r_6 = 27.$$

The unique optimal extreme point solution of $(P2)$ is

$$X_1 = (1\ 1\ 0\ 0\ 0\ 1) \quad \text{with} \quad R^1 = R(X_1) = 27.$$

Also, $F^1 = F_1(X_1) + F_2(X_1) = 30.$

$$R^1 < F^1.$$

The unique second best extreme point solution of (P2) is

$$X_2 = (0.5\ 0\ 0\ 0\ 0.5\ 1.5) \quad \text{with} \quad R^2 = (X_2) = 29.$$

Now, $F^2 = \text{Min }[F^1, F(X_2)] = F^1 = 30$, and $R^2 < F^2$. The set S_3 of third best extreme point solutions of $(P2)$ is $\{(1\,2\,0\,1\,0\,0), (0\,1\,0\,2\,1\,0), (0\,0\,0\,1\,1\,1)\}$, with $R^3 = 35$.

As $F^2 < R^3$, $F^2 = 30$ is the optimal value of $F(X)$ and the optimal feasible solution of $(P1)$ is $X_1 = (1\ 1\ 0\ 0\ 0\ 1)$, but X_1 is not an optimal feasible solution of $(P1')$. Now $R^2 < \text{Min }[F(X) : X \in \bigcup_{j=1}^{2} S_j, F(X) > 30] \le R^3$ therefore $\text{Min }[F(X) : X \in \bigcup_{j=1}^{2} S_j, F(X) > 30] = 35$ is the second best vaue of $F(X)$ and the second best feasible solution is $X_2 = (0.5\ 0\ 0\ 0\ 0.5\ 1.5)$. Also, X_2 satisfies the constraint $3x_1^2 + 4x_2^2 + 3x_3 + x_4 + 5x_5 + x_6 \le 7$, and is hence optimal for $(P1')$.

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