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A Bilevel Linear Programming Problem with Bottleneck Objectives

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Abstract

A bilevel programming problem with linear constraints is studied in this paper. The outer level objective function of the problem is a sum of two concave bottleneck functions, while the inner level objective function is a single concave bottleneck function, both of which are to be minimized. The problem is a non-convex programming problem. It is shown that an optimal solution to the problem is attainable at an extreme point of the underlying region. The extreme points of the given region are ranked in increasing order of the values of the outer objective function until that one is reached which is feasible for the problem. This point is then the required global optimal solution.

1. INTRODUCTION

Considerable amount of work is being done currently in the field of bilevel programming, and, more generally, in multilevel programming problems. Interest has been generated in this field not only because of the possibilities of its applications in many diverse areas, but also because of the mathematical complexities arising in these problems, which are not generally encountered in familiar single level mathematical programming problems.

Multilevel programming problems occur in hierarchical administrative structures where there are decision makers at different levels of seniority, but no decision maker has direct control over the decisions of the others. At the same time, a decision made by one decision maker influences the choice set of the others. Decisions are thus made independently, yet interactively, to arrive at an optimal policy. Multilevel programming problems arise, for instance, in agriculture where numerous producers receive policy signals regarding price supports, subsidies, etc., from the government and in turn make production decisions which affect the policy objective of the government.

The multilevel programming problem was first introduced by Kornai and Liptak [38] in the context of resource allocation among independently working sectors coordinated by a central planning agency. A formal definition of the problem was given by Candler and Townsley [24].

Some of the applications of multilevel programming are problems pertaining to government involving issues such as the setting of penalties for illegal drug import, the fixing of import quotas and the development of transportation and communications infrastructure [24]. Applications to strategic weapons exchange problems and to the distribution of federal budgets among states have been described respectively by Bracken et al. [22] and Cassidy et al. [25]. Anandalingam and Apprey [7] have given a new approach to conflict resolution based on multilevel mathematical programming, and have illustrated it with a real world example of the solution of a water conflict problem between India and Bangladesh.

The special case of the multilevel programming problem in which only two levels are involved, called the bilevel programming problem, can be thought of as a static version of the Stackelberg leader-follower game [47,48] in which a Stackelberg strategy is used by the leader (or the higher level decision maker), given the rational reaction of the follower (or the lower level decision maker). The bilevel programming structure has been used to model problems concerning organizational design [4], spatial competition [36], facility location [50], signal optimization [40] and traffic assignment [17]. A real-world bilevel programming model of the highway network design problem has been constructed by Ben-Ayed et al. [15].

Studies in the bilevel linear programming problem have shown that it is non-convex [20] and NP-hard [14]. Several methodologies have been put forward to solve the bilevel linear programming problem. In the Parametric Complementary Pivot (PCP) algorithm suggested by Bialas et al. [21], the inner problem is replaced by its equivalent K-T conditions and the value of the outer objective function increased parametrically, taking care of the complementary slackness conditions. However, the algorithm contains certain inadequacies as has been pointed out by Ben-Ayed and Blair [14]. Bard [10] has suggested a Grid Search Algorithm in which the problem is first reduced to a semi-infinite linear programming problem and Kuhn-Tucker theory then applied to it. But a counter example in which the algorithm does not converge to the optimal solution has been given by Haurie et al. [37] who have pointed out the errors in the formulation of the algorithm due to which this happens.

Algorithms which find local optimal solutions to the bilevel linear programming problem have been suggested by Candler and Townsley [24] and Bialas and Karwan [20]. These algorithms are however, unable to identify its global optimal solution. Ünlü [52] has used the relationship between the linear bilevel problem and the linear bicriteria problem to find an optimal solution to the former. The *k th best* algorithm of Bialas and Karwan [19] uses an extreme point search procedure to find a global optimal solution to the bilevel linear program.

Anandalingam et al. [5] have used simulated annealing and genetic algorithm based approaches to obtain global optimal solutions to the bilevel linear program. In a technique suggested by Önal [44], the follower's problem is replaced by its equivalent Kuhn-Tucker conditions and the complementary slackness equations are moved to the leader's objective as a penalty function. The global optimal of the resulting quadratic programming problem is then reached by iterative application of a modified simplex algorithm.

Anandalingam and White [6] and White and Anandalingam [54] have developed a solution methodology for the bilevel programming problem using a duality gap - penalty function format. An exact penalty function is shown to exist for obtaining its global optimal solution. For each penalty parameter value, the central optimization problem is one of maximizing a convex function over a polytope, for which a modification of an algorithm by Tuy [51] is used. It has also been shown that this methodology can be extended to solve certain nonlinear bilevel problems. Shimizu and Aiyoshi [46] and Aiyoshi and Shimizu [2] have also used a penalty function approach for solving nonlinear bilevel programming problems.

Some other notable studies have also been made of bilevel nonlinear programming and multilevel programming problems in general. Benson [16] has studied the structure and properties of the linear multilevel programming problem, while Bard [11] has examined the convex two-level optimization problem. A detailed application and solution of a three-level programming problem has been described by Cassidy et al. [25], and a hybrid algorithm for the three-level linear programming problem has been proposed by Wen and Bialas [53]. Bard, Moore and Edmunds [12, 27, 42] have contributed to the work on the discrete, the mixed integer and the mixed integer nonlinear bilevel programming problems. Anandalingam [4] has extended the bilevel linear program to a decentralized bilevel system and to the hierarchical system with many levels of decision makers.

This paper introduces a new kind of bilevel programming problem in which the objective functions at both levels involve the bottleneck function. The bottleneck function was first studied by Hammer [35] in the context of time minimizing transportation problems. In a transportation problem, if x_{ij} denotes the number of units of goods being transported from the i th source to the j th destination and t_{ij} is the time taken to do so (irrespective of the quantity of the goods), then the time taken on a particular route (i, j) is given by the function

$$\begin{aligned} t_{ij}(x_{ij}) &= t_{ij}, \quad \text{if } x_{ij} > 0, \\ &= 0, \text{ otherwise.} \end{aligned}$$

Assuming that all carriers start simultaneously on their respective routes, the time taken for completion of the transportation schedule is given by

$$T(x) = \underset{(i,j)}{\text{Max}} \left\{ t_{ij} \left(x_{ij} \right) \right\}.$$

This function $T(X)$ is called the bottleneck function, with the route taking maximum time for completion being called the bottleneck route for obvious reasons.

The time minimizing assignment problem also uses the bottleneck function. In an assignment problem the variable x_{ij} takes value 1 if the i th person is assigned to the j th job, and 0 otherwise; while t_{ij} is the time taken by the i th person to do the j th job.

Time minimizing transportation and assignment problems have been studied by many authors [1, 26, 29, 33, 49] while Bhatia et al. [18], Geetha and Vartak [30], Glickman and Berger [32] and others have studied their time-cost trade-off aspect.

The general bottleneck linear programming problem has been studied by Akgül [3], Frieze [28], Gupta et al. [34] and Seshan et al. [45]. Bansal et al. [9] have given a procedure for ranking of solutions of the bottleneck linear programming problem and also to find alternate solutions corresponding to each k th ($k \geq 1$) best value of the bottleneck objective function. Burkard and Rendl [23] have studied the lexicographic bottleneck programming problem. Minoux [41] has given many applications and studied the problem in which the objective function is the sum of a linear part and a bottleneck part. Geetha and Nair [31] have studied a similar situation as a variation of the assignment problem.

The problem being studied in this paper is of the form

$$\underset{X}{\text{Min}} \ F(X, Y), \tag{P1}$$

where Y solves

$$\underset{Y}{\text{Min}} \ (T(X, Y)/X), \tag{P2}$$

subject to

$$AX + BY = b,$$

$$X, Y \geq 0,$$

where $F(X, Y) = G(X, Y) + H(X, Y)$, and $G(X, Y)$, $H(X, Y)$ and $T(X, Y)$ are concave bottleneck functions whereas the bilevel linear programming problem is

$$\min_{X} \quad (aX + bY), \qquad (BLPP)$$

where Y solves

$$\min_{Y} \quad (cX + dY),$$

subject to

$$AX + BY = b,$$

$$X \geq 0, \; Y \geq 0.$$

To give one example of the practical importance of the bilevel bottleneck programming problem (P 1) being studied, consider a situation in which two categories of products are being produced, say, $N(1)$ types of primary products and $N(2)$ types of secondary products. The secondary products are required as inputs for the primary ones, and are produced by various local producers, while the primary products are processed centrally and in two consecutive, non-overlapping phases.

Let x_i be the number of units produced of the i th $(1 \leq i \leq N(1))$ primary product and y_j be the number of units produced of the j th $(1 \leq j \leq N(2))$ secondary product. Let $T(X, Y)$ be the time taken for production at the secondary level, and $G(X, Y)$ and $H(X, Y)$ be the times of the two phases of production at the primary level. Let $AX + BY = b$, $X \geq 0$, $Y \geq 0$ be the resource constraints for the various resources involved. It is assumed that atleast some of the resources are required at both levels of production, so that the local producers have to make adjustments according to the decision of the primary producer. This situation fits into the structure of the problem (P 1) being studied.

It has been proved in this paper that an optimal solution to (P 1) is attainable at an extreme point of the region $S = \{(X, Y) \in R^{N(1)} \times R^{N(2)} / AX + BY = b, X, Y \geq 0\}$ (which is assumed to be regular), and hence the extreme point search algorithm seems to be suitable for finding a global optimal solution to (P 1).

The authors have, as yet, not been able to make successful use of the penalty function method, the modified simplex approach of Önal [44] or any of the other methodologies referred to earlier, because of the very different nature of the bottleneck functions appearing in the two objective functions in (P 1).

The extreme point search algorithm developed for (P 1) has been described in Section 5. Sections 3 and 4 establish the main results required for the development of the algorithm. The next section gives a detailed description of the problem being studied.

2. DETAILS OF THE PROBLEM

The bilevel bottleneck linear programming problem being studied in this paper is

$$\underset{X}{\text{Min}} \ F(X,Y), \tag{P 1}$$

where Y solves

$$\underset{Y}{\text{Min}} \ (T(X,Y)/X), \tag{P 2}$$

subject to

$(X,Y) \in S = \{(X, Y) \in R^{N(1)} \times R^{N(2)} / AX + BY = b, X, Y \geq 0\}$,

where $A \in R^{m \times N(1)}$, $B \in R^{m \times N(2)}$, $b \in R^m$, $N(1) + N(2) = n$,

rank (A) = rank (B) = m , and S is assumed to be regular.

Define the index sets $I = \{1, \ldots, N(1)\}$ and $J = \{1, \ldots, N(2)\}$. The functions $F(X,Y)$ and $T(X,Y)$ are defined as :

$$F(X,Y) = G(X,Y) + H(X,Y),$$

where

$$G(X,Y) = \text{Max}\left\{g_i(x_i),\ i \in I\ ;\ g_j'(y_j), j \in J\right\},$$

$$H(X,Y) = \text{Max}\left\{h_i(x_i),\ i \in I\ ;\ h_j'(y_j), j \in J\right\},$$

$$T(X,Y) = \text{Max}\left\{t_i(x_i),\ i \in I\ ;\ t_j'(y_j), j \in J\right\},$$

and for all $i \in I$, $g_i(x_i) = g_i$ if $x_i > 0$ $(g_i \in R^+)$

$= 0$ otherwise ,

and $g_j'(y_j)\ \forall\ j \in J$; $h_i(x_i)$, $t_i(x_i)\ \forall\ i \in I$; $h_j'(y_j)$, $t_j'(y_j)\ \forall\ j \in J$, are similarly defined. G, H and T are bottleneck functions. Clearly, they are concave in R^n [9].

For a better understanding of the problem being studied, some observations and remarks are made below. The problem is seen to have many similarities with the conventional bilevel linear programming problem, as also certain significant differences (see Remarks 5,6 and 8) which arise because of the bottleneck functions. The set of feasible solutions of $(P\,1)$ is very different from that of the bilevel linear program, yet the optimal solution for both occurs at an extreme point of S. The following remarks and observations aim at comparing and contrasting $(P\,1)$ with $(BLPP)$.

Some Remarks and Observations

(a) The problems $(P1)$ and $(P2)$ are called the outer and inner problems respectively. The vectors X and Y are called the policy and behavioural vectors respectively and the corresponding variables are called policy (or outer) and behavioural (or inner) variables [24]. $(P1)$ and $(P2)$ are also called the leader's and the follower's problem respectively, and X and Y are correspondingly called the leader's and the follower's decision variables [47].

(b) For a given $\bar{X} \in R^{N(1)}$, the feasible region for $(P2)$ is given by $S_{\bar{X}} = \{Y \in R^{N(2)} / BY = b - A\bar{X},\ Y \geq 0\}$. Thus the inner problem for $\bar{X} \in R^{N(1)}$ can also be written as $\underset{Y \in S_{\bar{X}}}{\text{Min}}\ T(\bar{X}, Y)$. Clearly, $S_{\bar{X}}$ is a convex region. $S_{\bar{X}}$ will be non-empty if there exists a $\bar{Y} \in R^{N(2)}$ such that $(\bar{X}, \bar{Y}) \in S$.

(c) The set $\{X \in R^{N(1)} / X \geq 0\}$ is called the policy space of $(P\,1)$. An $\bar{X}$ belonging to the policy space of $(P\,1)$ is called a feasible or an infeasible policy setting for $(P\,1)$ depending on whether $S_{\bar{X}}$ is non-empty or empty.

The set $\{X \in R^{N(1)} / X \geq 0,\ S_X \neq \phi\}$ is called the feasible policy space of $(P\,1)$ and clearly, it is the projection of S on the policy space. Also, the feasible policy space is a convex region. It may be noted that an optimal feasible solution of $(P\,1)$ need not necessarily correspond to an extreme point of the feasible policy space.

(d) The feasible region for $(P\,1)$ is

$$S_1 = \{(\bar{X}, \bar{Y}) \in S / \bar{X} \text{ is a feasible policy setting and } \bar{Y} \text{ solves } \underset{Y \in S_{\bar{X}}}{\text{Min}}\ T(\bar{X}, Y)\}$$

$$= \{(\bar{X}, \bar{Y}) \in S / S_{\bar{X}} \neq \phi,\ T(\bar{X}, \bar{Y}) = \underset{Y \in S_{\bar{X}}}{\text{Min}}\ T(\bar{X}, Y)\}.$$

Thus the outer problem may be written as $\underset{(X,\,Y) \in S_1}{\text{Min}}\ F(X, Y)$. Clearly, if

$S_{\bar{X}} \neq \phi$ and $\bar{Y}$ is a non-optimal feasible solution of the inner problem at $\bar{X}$, then $(\bar{X}, \bar{Y}) \notin S_1$.

(e) S_1 is, in general, not a convex set (the same is true in the linear case also [19]). S_1 is a union of disjoint convex sets, all of which are not necessarily closed. Also, S_1 is connected, in the sense that between any two points of S_1 there exists a piecewise linear continuous path lying entirely in S_1.

The following example illustrates the observations made here :

$$\min_{X} \; F(X, Y),$$

where Y solves

$$\min_{Y} \; (T(X, Y)/X),$$

and the region S is given by the constraints :

$$\begin{aligned}
x_1 + 4y_1 - y_2 &= 16 \\
x_1 + 4y_1 + y_3 &= 32 \\
-x_1 + y_1 + y_4 &= 3 \\
x_1 + 2y_1 + y_5 &= 22 \\
x_1 - y_1 + y_6 &= 10, \\
x_1 \geq 0, \; y_j \geq 0, \; j &= 1, \ldots, 6,
\end{aligned}$$

$t_1 = 7$, $t_1' = 13$, $t_2' = 24$, $t_3' = 30$, $t_4' = 28$, $t_5' = 20$, $t_6' = 17$.

The two open, hatched regions A_1 and A_2 (not including the dotted boundaries) in Fig. 1 (which depicts a mapping of the given region S onto the space R^2), along with A_3 constitute the set S_1 of feasible solutions of (P 1).

(f) The weak convexity like property of S_1, which holds in the linear case [19], may not hold in this case.

(g) An extreme point of S_1 is an extreme point of S.

(h) Uniqueness of optimal solution of the inner problem is not essential.

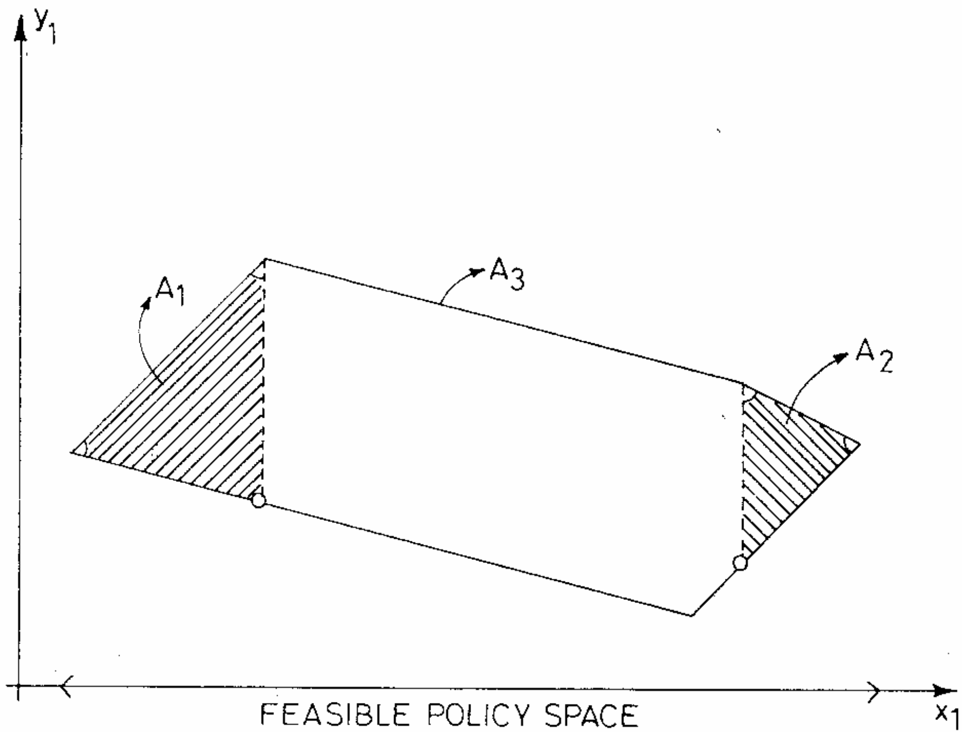

Fig. 1

Unlike linear bilevel programming, the correspondence between the set S_1 and the feasible policy space may not be 1-1.

(i) Clearly, if $(\bar{X}, \bar{Y}) \in \text{Int}(S)$, the set of interior points of S, then $\bar{Y} \in \text{Int}(S_{\bar{X}})$. Thus if $\bar{Y}$ is an extreme point of $S_{\bar{X}}$, then $(\bar{X}, \bar{Y}) \notin \text{Int}(S)$.

(j) The set $S^*_{\bar{X}} = \{Y \in R^{N(2)} / Y \text{ solves } \min_{Y \in S_{\bar{X}}} T(\bar{X}, Y)\}$ is a convex set.

For a specified $\bar{X} \in R^{N(1)}$, the minimum value of $T(\bar{X}, Y)$ is attainable at an extreme point of $S^*_{\bar{X}}$ [9, 13, 39].

(k) Every feasible solution of $(P\,1)$ is a feasible solution of the problem $\min_{(X,Y) \in S} F(X, Y)$. The optimal value of $F(X, Y)$ in $\min_{(X,Y) \in S} F(X, Y)$ is, in general, smaller than the optimal value of $F(X, Y)$ in $(P\,1)$.

If $(\bar{X}, \bar{Y})$ is an optimal basic feasible solution of $\min_{(X,Y) \in S} F(X, Y)$ with

$\bar{Y}$ being an optimal basic feasible solution of $\underset{Y \in S_{\bar{X}}}{\text{Min}} \; T(\bar{X}, Y)$ then $(\bar{X}, \bar{Y})$ is an optimal feasible solution of $(P\,1)$.

Assumptions

The following assumptions have been made throughout this paper :

(i) $N(2) > m$ and (ii) $t'_j > \max_{i \in I} \{t_i\} \;\forall\; j \in J$.

If $N(2) \leq m$, then for any $\bar{X} \in R^{N(1)}$, the system $BY = b - A\bar{X}$, $Y \geq 0$ has atmost one solution, due to which the inner problem is no longer meaningful. The second condition ensures that for any $\bar{X} \in R^{N(1)}$, the value of $T(\bar{X}, Y)$ is controlled by Y and so $T(\bar{X}, Y)$ does not have the same value on the entire feasible region $S_{\bar{X}}$ of the corresponding inner problem. In other words, the policy variables control only the feasibility of the inner problem and not its optimality. This condition is necessitated due to the bottleneck objective functions.

Thus both the assumptions made here are necessary to retain the essential character of the bilevel problem being studied.

3. SOME RESULTS

Theorem 1. *Let $(\bar{X}, \bar{Y})$ be a non-extreme point of S lying on one of its edges or facets. If $(\bar{X}, \bar{Y})$ is feasible for $(P\,1)$, then there exists atleast one extreme point of S on that edge or facet which is also feasible for $(P\,1)$.*

Proof. Since $(\bar{X}, \bar{Y})$ is not an extreme point of S, there exist extreme points (X^k, Y^k), $k = 1, \ldots, p$ of S lying on the edge or facet [43] such that

$$(\bar{X}, \bar{Y}) = \sum_{k=1}^{p} \lambda_k (X^k, Y^k), \; \sum_{k=1}^{p} \lambda_k = 1, \lambda_k > 0, \;\forall\; k. \tag{1}$$

By definition of the bottleneck function T and as $\lambda_k > 0$, $\forall\; k$,

$$T(\bar{X}, \bar{Y}) = \underset{1 \leq k \leq p}{Max} \; T(X^k, Y^k). \tag{2}$$

Assume that none of the points (X^k, Y^k), $k = 1, \ldots, p$ is feasible for $(P\,1)$. This implies that Y^k is not optimal for $\underset{Y \in S_{X^k}}{\text{Min}} \; T(X^k, Y)$ for each k.

Let $\hat{Y}^k \in S_{X^k}$ $(k = 1, \ldots, p)$ be an optimal solution of $\underset{Y \in S_{X^k}}{\text{Min}} \; T(X^k, Y)$

Then $\quad T(X^k, \hat{Y}^k) < T(X^k, Y^k), \; \forall \; k = 1, \ldots, p.$ (3)

As $\hat{Y}^k \in S_{X^k}$, $(X^k, \hat{Y}^k) \in S$, $\forall \; k = 1, \ldots, p$.

S being convex, $\sum_{k=1}^{p} \lambda_k (X^k, \hat{Y}^k) \in S$, i.e.,

$$(\bar{X}, \hat{Y}) \in S, \quad \text{where } \hat{Y} = \sum_{k=1}^{p} \lambda_k \hat{Y}^k. \tag{4}$$

Therefore, $\hat{Y} \in S_{\bar{X}}$.

But $\bar{Y}$ is an optimal solution for $\underset{Y \in S_{\bar{X}}}{\text{Min}} \; T(\bar{X}, Y)$, since $(\bar{X}, \bar{Y})$ is feasible for $(P\,1)$.

Therefore, $\quad T(\bar{X}, \bar{Y}) \le T(\bar{X}, \hat{Y})$. (5)

Now $T(\bar{X}, \hat{Y}) = \underset{1 \le k \le p}{\text{Max}} \; T(X^k, \hat{Y}^k)$, by (1) and (4)

$< \underset{1 \le k \le p}{\text{Max}} \; T(X^k, Y^k)$ from (3)

$= T(\bar{X}, \bar{Y})$ from (2). (6)

(5) and (6) contradict each other. Therefore, the assumption that none of the points (X^k, Y^k) $(1 \le k \le p)$ is feasible for $(P\,1)$ is wrong. Hence if $(\bar{X}, \bar{Y})$ is feasible for $(P\,1)$, there exists atleast one extreme point of S such that it is feasible for $(P\,1)$.

The result stated in the next theorem, although similar to the one in the standard bilevel linear programming problem, is important to prove as it pertains to the minimization of the sum of two concave bottleneck functions (as opposed to the minimization of a linear function in (BLPP) over S_1, which is a connected non-convex region.

Theorem 2. *The optimal value of F(X,Y) in $(P\,1)$ is attainable at an extreme point of S.*

Proof. Let $(\bar{X}, \bar{Y})$ be an optimal solution of $(P\,1)$. Assume that $(\bar{X}, \bar{Y})$ is not an extreme point of S. Further assume that $(\bar{X}, \bar{Y})$ is a point on a facet of S. (The possibility of $(\bar{X}, \bar{Y})$ being an interior point of S can be safely ignored, because an optimal solution of $\underset{Y \in S_{\bar{X}}}{\text{Min}}\ T(\bar{X}, Y)$ is always attainable at an extreme point of $S_{\bar{X}}$, $T(\bar{X}, Y)$ being concave [9, 39]. As remarked earlier, an extreme point of $S_{\bar{X}}$ corresponds to a point on a facet of S).

By previous theorem, there exists an extreme point, say $(\hat{X}, \hat{Y})$ on the facet which is also feasible for $(P\,1)$.

Let
$$(\bar{X}, \bar{Y}) = \hat{\lambda}\,(\hat{X}, \hat{Y}) + \sum_{i=1}^{l} \lambda_i\,(X^i, Y^i),\ \hat{\lambda} + \sum_{i=1}^{l} \lambda_i = 1,$$
$$0 < \hat{\lambda} < 1,\ 0 \le \lambda_i \le 1,\ \forall\ i = 1, \ldots, l, \tag{7}$$

where (x^i, y^i), $i = 1, \ldots, l$ are other extreme points on the facet. Since $(\bar{X}, \bar{Y})$ is optimal for $(P\,1)$ and $(\hat{X}, \hat{Y})$ is feasible for $(P\,1)$, therefore
$$F(\bar{X}, \bar{Y}) \le F(\hat{X}, \hat{Y}). \tag{8}$$

Now by nature of the bottleneck function and by (7), $G(\bar{X}, \bar{Y}) \ge G(\hat{X}, \hat{Y})$ and $H(\bar{X}, \bar{Y}) \ge H(\hat{X}, \hat{Y})$, which implies
$$F(\bar{X}, \bar{Y}) \ge F(\hat{X}, \hat{Y}). \tag{9}$$

(8) and (9) give $F(\bar{X}, \bar{Y}) = F(\hat{X}, \hat{Y})$. Thus the optimal value of $F(X, Y)$ in $(P1)$ is attainable at an extreme point of S. The following result gives a method for testing the feasibility of a given point in S for the problem $(P\,1)$.

Theorem 3. *Let $(\bar{X}, \bar{Y}) \in S$. Then $(\bar{X}, \bar{Y})$ is feasible for $(P\,1)$ if the optimal value of $Z(Y)$ is non-zero and finite in the linear programming problem*

$$LP(\bar{T}/\bar{X}) : \underset{S_{\bar{X}}}{\text{Min}}\ Z(Y) = \sum_{j=1}^{N(2)} C_j Y_j,$$

where
$$C_j = 0,\ t_j' < \bar{T}$$
$$= 1,\ t_j' = \bar{T}$$
$$= \infty,\ t_j' > \bar{T},\ j = 1, \ldots, N(2),$$

and $\bar{T} = T(\bar{X}, \bar{Y})$.

Proof. $(\bar{X}, \bar{Y})$ is feasible for $(P\,1)$ iff $\bar{Y}$ solves $(P\,2)$ for $X = \bar{X}$. Assume that $\bar{Y}$ is not an optimal solution of $(P\,2)$ for $X = \bar{X}$. Then, there exists a $\hat{Y} \in R^{N(2)}$

such that $\hat{Y} \in S_{\bar{X}}$ and $T(\bar{X}, \hat{Y}) < T(\bar{X}, \bar{Y}) = \bar{T}$. This implies $Z(\hat{Y}) = 0$, which contradicts the fact that the optimal value of $Z(Y)$ in $LP(\bar{T}/\bar{X})$ is nonzero. Therefore $(\bar{X}, \bar{Y})$ must be feasible for (P 1).

Now that it is established that the optimal value of $F(X, Y)$ in (P 1) can be obtained at an extreme point of S (Theorem 2), it makes sense to develop a procedure in which the extreme points of S are systematically searched. This is done by ranking the extreme points of S in ascending order of values of $F(X,Y)$ and testing each point for feasibility of (P 1). Ranking is stopped the moment an extreme point is reached which passes the feasibility test.

The next section discusses the problem $\underset{S}{\text{Min}}\ F(X, Y)$ along with a procedure to rank its extreme point solutions.

4. THE PROBLEM $\underset{S}{\text{Min}}\ F(X, Y)$

Consider the problem

$$\underset{S}{\text{Min}}\ F(X, Y), \qquad (P\ 3)$$

where $F(X,Y) = G(X,Y) + H(X,Y)$, and the functions $G(X,Y)$ and $H(X,Y)$ are defined before. Ranking of extreme point solutions of (P 3) will be done with the help of a related bottleneck linear programming problem :

$$\underset{S}{\text{Min}}\ R(X, Y), \qquad (P\ 4)$$

where $\quad R(X, Y) = \text{Max}\{r_i(x_i), i \in I\ ;\ r_j'(y_j), j \in J\},$

and $\quad r_j(x_i) = g_i(x_i) + h_i(x_i), \forall\ i \in I,$

$$r_i'(y_j) = g_j'(y_j) + h_j'(y_j), \forall\ j \in J,$$

It may be noted that the objective function in (P 4) is a single concave bottleneck function and hence attains its minimum value at an extreme point of the convex feasible region S [9,39]. The next five results will be required to develop an algorithm to find an optimal solution to (P 3).

Theorem 4. *Every k th best value of $R(X, Y)$ is attainable at an extreme point of S.*

Proof. Let R^k be the $k\,th$ best value of $R(X, Y)$ and $R(\bar{X}, \bar{Y}) = R^k$.

Suppose $(\bar{X}, \bar{Y})$ is not an extreme point of S. We can safely ignore the possibility of $(\bar{X}, \bar{Y})$ being an interior point (since an interior point always gives the maximum possible value of R(X,Y)), so that $(\bar{X}, \bar{Y})$ is a point on a facet, say $\bar{S}$ of S. Thus

$$(\bar{X}, \bar{Y}) = \sum_{i=1}^{s} \mu_i (x^i, y^i), \ \sum_{i=1}^{s} \mu_i = 1, \ 0 < \mu_i < 1, \quad \forall \ i = 1, \ldots, s,$$

where (X^i, Y^i) $(1 \le i \le s)$ are extreme points on $\bar{S}$. By definition of the bottleneck function and since $0 < \mu_i < 1$, $\forall\ i$, $R(\bar{X}, \bar{Y}) = \underset{1 \le i \le s}{\text{Max}} R(X^i, Y^i)$ $= R(X^l, Y^l)$ (say) $(1 \le l \le s)$. Thus the kth best value R^k of $R(X, Y)$ can also be attained at an extreme point $(\hat{X}, \hat{Y})$ $(= (X^l, Y^l))$ of S.

An important result which will be frequently used is given in the following theorem :

Theorem 5. $R(X, Y) \le F(X, Y)$, $\forall$ $(X, Y) \in S$.

Proof.

$$\begin{aligned} R(X, Y) &= \text{Max}\{r_i(x_i), i \in I ; r_j'(y_j), j \in J\} \\ &= r_k(x_k) \quad \text{(say)} \\ &= g_k(x_k) + h_k(x_k) \\ &\le \text{Max}\{g_i(x_i), i \in I; g_j'(y_j), j \in J\} \\ &\qquad + \text{Max}\{h_i(x_i), i \in I; h_j'(y_j), j \in J\} \\ &= G(X, Y) + H(X, Y) \\ &= F(X, Y), \forall (X, Y) \in S. \end{aligned}$$

Thus, it is seen that the objective function in (P 4) provides a lower bound for the objective function in (P 3) over the whole of S. This fact, along with ranking of values of $R(X, Y)$ will be used to rank the extreme point solutions of (P 3).

Notations

S^k : set of kth best extreme point solutions of (P4).

$R^k = R(X, Y), (X, Y) \in S^k$.

(X^k, Y^k) : a point in S^k such that

$$F(X^k, Y^k) = \text{Min}\{F(X, Y) : (X, Y) \in S^k\}.$$

$(\bar{X}^k, \bar{Y}^k)$: an element in $\{(X^1, Y^1), \ldots, (X^k, Y^k)\}$ such that

$$F(\bar{X}^k, \bar{Y}^k) = \text{Min}\{F(X, Y) : (X, Y) \in \bigcup_{i=1}^{k} S^i\}$$

$$= \text{Min}\{F(X^1, Y^1), \ldots, F(X^k, Y^k)\}.$$

Theorem 6. *If $R^1 = F(X^1, Y^1)$, then (X^1, Y^1) is an optimal solution of $(P\,3)$.*

Proof. $F(X^1, Y^1) = R^1$

$$\leq R(X, Y) \;\forall\; (X, Y) \in S, \text{ since } (X^1, Y^1) \in S^1$$

$$\leq F(X, Y) \;\forall\; (X, Y) \in S.$$

Hence (X^1, Y^1) is an optimal solution for $(P\,3)$.

Theorem 7. *If $F(\bar{X}^k, \bar{Y}^k) \leq R^{k+1}$ for some $k \geq 1$, then $(\bar{X}^k, \bar{Y}^k)$ is an optimal solution for $(P\,3)$.*

Proof. By definition of $(\bar{X}^k, \bar{Y}^k)$,

$$F(\bar{X}^k, \bar{Y}^k) \leq F(X, Y), \;\forall\; (X, Y) \in \bigcup_{i=1}^{k} S^i. \tag{10}$$

By hypothesis

$$F(\bar{X}^k, \bar{Y}^k) \leq R^{k+1}$$

$$= R(X^{K+1}, Y^{k+1})$$

$$\leq F(X^{k+1}, Y^{k+1}) \quad \text{(Theorem 5)}$$

$$= F(X, Y) \;\forall\; (X, Y) \in S^{k+1}$$

(by definition of (X^{k+1}, Y^{k+1}))

Therefore, $F(\bar{X}^k, \bar{Y}^k) \leq F(X, Y), \forall\; (X, Y) \in S^{k+1}$ (11)

Again, by hypothesis

$$F(\bar{X}^k, \bar{Y}^k) \leq R(X^{k+1}, Y^{K+1})$$

$$< R(X, Y) \forall (X, Y) \in S \setminus \bigcup_{i=1}^{k+1} S^i$$

$$\leq F(X, Y) \qquad \text{(Theroem 5)}$$

Therefore, $F(\bar{X}^k, \bar{Y}^k) < F(X, Y) \forall (X, Y) \in S \setminus \bigcup_{i=1}^{k+1} S^i$. (12)

(10), (11) and (12) imply that no extreme point solution of S other than $(\bar{X}^k, \bar{Y}^k)$ can yield a better value of $F(X, Y)$, i.e., $F(\bar{X}^k, \bar{Y}^k) \leq F(X, Y) \forall (X, Y) \in S$. Hence $(\bar{X}^k, \bar{Y}^k)$ is an optimal solution for $(P\,3)$.

Theorem 8. For $k \geq 2$, if $R^k = F(\bar{X}^k, \bar{Y}^k)$ *then* $(\bar{X}^k, \bar{Y}^k)$ *is optimal for (P 3).*

Proof. By hypothesis

$$F(\bar{X}^k, \bar{Y}^k) = R^k$$

$$= R(X^k, Y^k)$$

$$= R(X, Y) \forall X \in S \setminus \bigcup_{i=1}^{k} S^i$$

$$\leq F(X, Y) \qquad \text{(Theorem 5)}$$

Therefore, $F(\bar{X}^k, \bar{Y}^k) < F(X, Y) \forall (X, Y) \in S \setminus \bigcup_{i=1}^{k} S^i$. (13)

Also, by definition of $(\bar{X}^k, \bar{Y}^k)$,

$$F(\bar{X}^k, \bar{Y}^k) \leq F(X, Y) \forall (X, Y) \in \bigcup_{i=1}^{k} S^i. \qquad (14)$$

(13) and (14) imply $F(\bar{X}^k, \bar{Y}^k) \leq F(X, Y), \forall (X, Y) \in S$.

Algorithm 1 : To find the best extreme point solution of $(P\,3)$

Step 1. Find the set S^1 of optimal basic feasible solutions of $(P\,4)$ [9]. Find (X^1, Y^1).

If $R^1 = F(X^1, Y^1)$, (X^1, Y^1) is an optimal solution of $(P3)$.

If $R^1 < F(X^1, Y^1)$, set $k = 1$, $(\bar{X}^1, \bar{Y}^1) = (X^1, Y^1)$ and go to Step 2.

Step 2. Find the set S^{k+1} of $(k+1)$th best basic feasible solutions of $(P4)$ [9]. Find R^{k+1}.

If $F(\bar{X}^k, \bar{Y}^k) \leq R^{k+1}$, $(\bar{X}^k, \bar{Y}^k)$ is an optimal solution of $(P3)$.

If $R^{k+1} < F(\bar{X}^k, \bar{Y}^k)$, go to Step 3.

Step 3. Find $(\bar{X}^{k+1}, \bar{Y}^{k+1})$.

If $R^{k+1} = F(\bar{X}^{k+1}, \bar{Y}^{k+1})$, $(\bar{X}^{k+1}, \bar{Y}^{k+1})$ is an optimal solution of $(P3)$.

If $R^{k+1} < F(\bar{X}^{k+1}, \bar{Y}^{k+1})$, go to Step 2 for next higher value of k.

Algorithm 2 : To find all the kth $(k \geq 1)$ best extreme point solutions of $(P3)$.

The following additional notations have been used :

S_r^* : set of rth best extreme point solutions of $(P3)$.

(X_r^*, Y_r^*) : an rth best extreme point solution of $(P3)$.

R^L: largest value of $R(X, Y)$.

S^L: set of extreme point solutions of (P4) yielding value R^L.

Clearly, $R^L = \text{Max}\{R(X, Y) : (X, Y) \in S\} = R(X, Y)$, $(X, Y) \in S^L$.

Let $(X_1^*, Y_1^*) = (\bar{X}^t, \bar{Y}^t)$ for some $t \geq 1$, where

$$F(\bar{X}^t, \bar{Y}^t) = \text{Min}\{F(X, Y) : (X, Y) \in \bigcup_{i=1}^{t} S^i\}.$$

Step 1. Set $k = t$, $r = 2$ and $S_1^* = \{(\bar{X}^t, \bar{Y}^t)$

Step 2. If $k < L$, go to Step 3.

If $k = L$, go to Step 7.

Step 3. Set $\hat{S} = \bigcup_{i=1}^{k} S^{i} \setminus S^{*}_{r-1}$.

If $\hat{S} = \phi$, find S^{k+1}, R^{k+1} and return to Step 3 for next higher value of k.

If $\hat{S} \neq \phi$, find $(\bar{X}, \bar{Y}) \in \hat{S}$ such that

$F(\bar{X}, \bar{Y}) = \text{Min}\ \{F(X, Y) : (X, Y) \in \hat{S}\ \}$ and go to Step 4.

Step 4. If $F(\bar{X}, \bar{Y}) = F(X^{*}_{r-1}, Y^{*}_{r-1})$, then set

$S^{*}_{r-1} = S^{*}_{r-1} \cup \{(\bar{X}, \bar{Y})\}$ and go to Step 3.

If $F(\bar{X}, \bar{Y}) > F(X^{*}_{r-1}, Y^{*}_{r-1})$, go to Step 5.

Step 5. If $F(\bar{X}, \bar{Y}) \leq R^{k}$, then $(\bar{X}, \bar{Y}) \in S^{*}_{r}$.

Set $S^{*}_{r} = \{(\bar{X}, \bar{Y})\}$, $r = r + 1$ and go to Step 3.

If $F(\bar{X}, \bar{Y}) > R^{k}$, go to Step 6.

Step 6. Find R^{k+1}

If $F(\bar{X}, \bar{Y}) \leq R^{k+1}$, set $S^{*}_{r} = \{(\bar{X}, \bar{Y})\}$.

Set $k = k + 1$, $r = r + 1$ and go to Step 3.

If $F(\bar{X}, \bar{Y}) > R^{k+1}$, set $k = k + 1$, go to Step 3.

Step 7. If $\{(X, Y) : (X, Y) \in \bigcup_{i=1}^{L} S^{i} \setminus \bigcup_{i=1}^{r-1} S^{*}_{i}\ \} = \phi$, stop.

All the extreme points of S have been ranked.

If $\{(X, Y) : (X, Y) \in \bigcup_{i=1}^{L} S^{i} \setminus \bigcup_{i=1}^{r-1} S^{*}_{i}\ \} \neq \phi$, find

$\text{Min}\ \{ F(X, Y) : (X, Y) \in \bigcup_{i=1}^{L} S^{i} \setminus \bigcup_{i=1}^{r-1} S^{*}_{i}\ \} = F(\bar{X}, \bar{Y})$ (*say*).

If $F(\bar{X}, \bar{Y}) = F(X^*_{r-1}, Y^*_{r-1})$, then set $S^*_{r-1} = S^*_{r-1} \cup \{(\bar{X}, \bar{Y})\}$ and return to Step 7.

If $F(\bar{X}, \bar{Y}) > F(X^*_{r-1}, Y^*_{r-1})$, set $S^*_r\{(\bar{X}, \bar{Y})\}$ and repeat Step 7 for next higher value of r.

5. ALGORITHM TO FIND AN OPTIMAL SOLUTION OF $(P1)$.

The two algorithms developed in the previous section, along with the feasibility test in Theorem 3, are now used to fomulate an algorithm to find an optimal solution to $(P1)$.

Step 1. Find S^*_1. $S^*_1 \neq \phi$, since S is regular.

Choose some $(\bar{X}, \bar{Y}) \in S^*_1$. Let $T(\bar{X}, \bar{Y}) = \bar{T}$. Let the optimal solution of $LP(\bar{T}/\bar{X})$ be $\hat{Y}$.

If $Z(\hat{Y})$ is non-zero and finite, go to Step 4.

If $Z(\hat{Y}) = 0$, set $W = \{(\bar{X}, \bar{Y})\}$ and go to Step 3 for $i = 1$.

Step 2. Find the set S^*_i of the ith best basic feasible soutions of $(P3)$ (ref. alg. 2), and go to Step 3.

Step 3. If $\left(\bigcup_{u=1}^{i} S^*_u \setminus W \right) = \phi$, set $i = i + 1$ and return to Step 2.

Otherwise, choose some $(\bar{X}, \bar{Y}) \in \bigcup_{u=1}^{i} S^*_i \setminus W$. Let $T(\bar{X}, \bar{Y}) = \bar{T}$.

Solve the linear programming problem $LP(\bar{T}/\bar{X})$. Let its optimal solution be $\hat{Y}$.

If $Z(\hat{Y})$ is non-zero and finite, go to Step 4.

If $Z(\hat{Y}) = 0$, set $W = W \cup \{(\bar{X}, \bar{Y})\}$ and return to Step 3.

Step 4. $(\bar{X}, \bar{Y})$ is an optimal solution to $(P1)$.

6. CONCLUDING REMARKS

(i) Algorithm 1 aims at finding the optimal feasible solution of the problem $(P3)$, which involves minimization of the sum of two concave bottleneck functions over the polytope S. To the best of the authors' knowledge, no other algorithm for solving (P 3) exists. Algorithm 1 requires ranking of the extreme point solutions of the problem (P 4). Although the extreme point ranking procedure used is known to be inefficient it seems to be the only way out in the present situation.

It may also be noted that Algorithm 2, for ranking the extreme point solutions of (P 3), is distinctly different from the k th best algorithm [19], and this is because of the presence of the bottleneck functions in (P 3).

(ii) As the extreme point solutions of (P 3) are ranked in increasing order of values of $F(X, Y)$ until such a point is reached which is feasible for (P 1), it follows that this terminal extreme point yields the global minimum value of $F(X, Y)$ over S_1, the feasible region for (P 1).

(iii) It may be noted that the best extreme point solution and subsequent extreme point solutions of (P 3) are obtained by systematic ranking of the extreme point solutions of (P 4) in increasing order of the values of $R(X, Y)$. As there is no order relation between the ranks of the extreme point solutions of (P 3) and (P 4), it becomes necessary to scan the extreme points of S with respect to (P 4), read the value of $F(X, Y)$ at these points and then rank them for (P 3).

As extreme point ranking techniques are computer intensive, the authors are currently trying to devise some other efficient approaches which involve bottleneck functions. They have not been successful in this endeavour so far, but hope that new and efficient algorithms will soon emerge.

(iv) If in (P 1), the function $F(X, Y)$ is given as

$$F(X, Y) = G(X, Y).H(X, Y),$$

then the above algorithm will also solve the new problem.

The related linear programming problem in this case will be

$$\underset{S}{\text{Min}}\, R(X, Y),$$

$$R(X, Y) = \text{Max}\ \{r_i(x_i),\ i \in I;\ r_j'(y_j),\ J \in J\},$$

where $\quad r_i(x_i),\ g_i(x_i)\ h_i\ (x_i),\ i \in I,$

and $\quad r_j'(y_j),\ g_j'(y_j)\ h_j'\ (y_j),\ j \in J.$

Extensions to the cases when (i) $F(X, Y) = \sum_{i=1}^{m} G_i(X, Y)$ and (ii) $F(X, Y) = \prod_{i=1}^{m} G_i(X, Y)$, where $G_i(X, Y)$, $i = 1, \ldots, m$ are convave bottleneck functions, can also be easily carried out.

Example. Consider the problem $(P1)$ where the set S is given by the constraints

$x_1 + 2x_2 + y_1 + 2Y_2 + 3y_4 = 6,$

$3x_1 + x_2 + 2y_1 + y_3 + 2y_4 = 5,$

$x_1 + x_2 + y_1 + y_3 + y_4 = 3,$

$x_1, x_2 \geq 0, y_1, y_2, y_3, y_4 \geq 0$, and

$g_1 = 5, g_2 = 3, g_1' = 15, g_2' = 11, g_3' = 25, g_4' = 29,$

$h_1 = 27, h_2 = 11, h_1' = 19, h_2' = 9, h_3' = 7, h_4' = 10,$

$t_1 = 3, t_2 = 5, t_1' = 20, t_2' = 31, t_3' = 15, t_4' = 19.$

The related min-max problem is

$$\operatorname*{Min}_{S} R(X, Y), \tag{P4}$$

where $r_1 = 32, r_2 = 14, r_1' = 34, r_2' = 20, r_3' = 32, r_4' = 39.$

The set S^1 of optimal basic feasible solutions of $(P4)$ is $S^1 = \{(1\,2\,0\,1/2\,0\,0), (1\,0\,0\,5/2\;2\;0)\}$, with $(X^1, Y^1) = (1\,2\,0\,1/2\,0\,0)$, $F(X^1, Y^1) = 38$ and $R(X^1, Y^1) = 32$. Therefore, $R(X^1, Y^1) < F(X^1, Y^1)$.

The set S^2 of second best extreme point solutions of (P4) is $S^2 = \{(0\;1\;2\;1\;0\;0), (0\;0\;2\;2\;1\;0)\}$, with $(X^2, Y^2) = (0\,1\,2\,1\,0\,0)$, $R(X^2, Y^2) = 34$ and $F(X^2, Y^2) = 34$.

Now $R(X^2, Y^2) < F(X^1, Y^1)$.

Also $F(\bar{X}^2, \bar{Y}^2) = \text{Min}\{F(X^1, Y^1), F(X^2, Y^2)\} = 34 = R(X^2, Y^2)$.

Therefore, $(\bar{X}^2, \bar{Y}^2) = (X^2, Y^2)$ is an optimal solution of (P3).

Now, for the problem $\underset{Y \in S_{x^2}}{\text{Min}}\ T(\bar{X}^2, Y)$, the optimal solution is $\hat{Y} = (1\ 0\ 0\ 1)$, and $T(\bar{X}_2, \hat{Y}) = 20 < T(\bar{X}_2, \bar{Y}_2) = 31$. Therefore, $(\bar{X}^2, \bar{Y}^2)$ is not feasible for (P1).

Now $\hat{S} = (S^1 \cup S^2) \setminus \{\bar{X}^2, \bar{Y}^2)\}$

$= \{(1\ 2\ 0\ 1/2\ 0\ 0), (1\ 0\ 0\ 5/2\ 2\ 0), (0\ 0\ 2\ 2\ 1\ 0)\}.$

$F(\bar{X}, \bar{Y}) = \text{Min}\{F(X, Y) : (X, Y) \in \bar{S}\} = 38$ and

$(\bar{X}, \bar{Y}) = (1\ 2\ 0\ 1/2\ 0\ 0).$

The third best extreme point value of $R(X, Y)$ over S is 39, that is, $R^3 = 39$, with $S^3 = \{(2/3\ 5/3\ 0\ 0\ 0\ 2/3), (0\ 0\ 0\ 0\ 1\ 2), (0\ 1\ 1\ 0\ 0\ 1)\}$ So $F(\bar{X}, \bar{Y}) = 38 < R^3 = 39$. Hence $(\bar{X}, \bar{Y})$ is the second best extreme point solution of $(P3)$

Now, the optimal solution of $\underset{Y \in S_x}{\text{Min}}\ (T(\bar{X}, Y)/\bar{X})$ is $\hat{Y} = (0\ 1/2\ 0\ 0) = \bar{Y}$. Hence $(\bar{X}, \bar{Y})$ is feasible for $(P1)$. Thus $(\bar{X}, \bar{Y})$ is an optimal solution for $(P1)$.

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