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BICRITERIA BOTTLENECK LINEAR PROGRAMMING PROBLEM

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The bicriteria bottleneck linear programming problem is studied in this paper. The two criteria considered are concave bottleneck functions, and the feasible region is a non-empty convex polyhedron. It has been established that all the efficient pairs of values of the objective function can be attained at the extreme points of the feasible region. Based upon some related bottleneck linear programming problems and linear programming problems, a convergent algorithm to generate all the efficient pairs is proposed.

KEY WORDS Bicriteria programming problem, Bottleneck linear programming problem, non-linear programming, non-convex programming.

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1. INTRODUCTION

This paper studies the bicriteria bottleneck linear programming problem

$$\text{`Min'}_{X \in S}\ (F(X),\ T(X))$$

where $F(X)$ and $T(X)$ are concave bottleneck functions and S is the non-empty feasible region defined by linear constraints.

Bicriteria problems arise when two objectives of conflicting nature are to be optimized. Since these objectives do not attain their optimum values at the same point, a single optimal solution is not available. Instead, a range of efficient solutions involving trade-offs between the two objectives is obtained and the choice of solution left to the decision maker.

Bicriteria problems in which one of the objectives is of the bottleneck type and the other linear, have been studied by many authors [2, 4, 5, 8, 10, 15], and problems in which both objectives are linear have been dealt with in [1, 9, 14, 16].

The theoretical development required to find the efficient pairs of the bottleneck bicriteria linear programming problem is detailed in Section 2, while the algorithm is described in Section 3 with a simple example to demonstrate it.

2. THEORETICAL DEVELOPMENT

The problem under consideration is

$$\text{'Min'}_{X \in S} \, (F(X),\, T(X)) \tag{P}$$

where

$$F(X) = \operatorname*{Max}_j f_j(x_j), \qquad f_j(x_j) = f_j,\ x_j > 0 \qquad (f_j \in \mathbb{R}^+)$$
$$= 0,\ x_j = 0,$$
$$T(X) = \operatorname*{Max}_j t_j(x_j), \qquad t_j(x_j) = t_j,\ x_j > 0 \qquad (t_j \in \mathbb{R}^+)$$
$$= 0,\ x_j = 0,$$

and

$$S = \{X / X \in \mathbb{R}^n,\ AX = b,\ X \geq 0\}, \qquad A \in \mathbb{R}^{m \times n}, \qquad b \in \mathbb{R}^m.$$

It is assumed that S is non-empty. The functions $F(X)$ and $T(X)$ are concave and conflicting in nature, and attain their minimum values at distinct extreme points of S.

Some definitions

Given two solutions X and $\hat{X}$ of (P) in S, $\hat{X}$ is said to *dominate* over X (or, $\hat{X}$ is called a *better solution* of (P) than X) if any one of the following three conditions holds:

(i) $F(\hat{X}) < F(X)$ and $T(\hat{X}) < T(X)$
(ii) $F(\hat{X}) = F(X)$ and $T(\hat{X}) < T(X)$
(iii) $F(\hat{X}) < F(X)$ and $T(\hat{X}) = T(X)$.

If $\hat{X}$ dominates over X, the corresponding pair of values $(F(\hat{X}), T(\hat{X}))$ is also said to *dominate* over the pair $(F(X), T(X))$.

Efficient Solution. A solution $\hat{X}$ of (P) is said to be an *efficient solution* if it is not dominated by any other solution of (P).

Efficient Pair. For an efficient solution $\hat{X}$ of (P), the pair of values $(F(\hat{X}), T(\hat{X}))$ is called an *efficient pair.*

Theorem 1: *Every pair of efficient values of the objective function in* (P) *is attainable at an extreme point of S.*

Proof: Let X and Y be any two points of S. Clearly, $F(\lambda X + (1-\lambda)Y) = \text{Max}(F(X), F(Y))$ and $T(\lambda X + (1-\lambda)Y) = \text{Max}(T(X), T(Y))$ $\forall \lambda \in (0, 1)$. Therefore, $(F(X), T(X)) \leqq (F(\lambda X + (1-\lambda)Y),\ T(\lambda X + (1-\lambda)Y))$ and $(F(Y), T(Y)) \leqq (F(\lambda X + (1-\lambda)Y),\ T(\lambda X + (1-\lambda)Y))$ $\forall \lambda \in (0, 1)$, with the symbol $\leq$ holding good in at least one of the above inequalities when $(F(X), T(X)) \neq (F(Y), T(Y))$. (If $a = (a_i)$, $b = (b_i)$, $i \in I$, then $a \leq b$ implies $a_i < b_i$ for at least one i in I, while $a \leqq b$ allows the possibility $a = b$). This implies that an efficient pair is always attainable at an extreme point of S, since a non-extreme point can be expressed as a convex combination of two points of S, with at least one of these being an extreme point [13].

Remark 1: If X_1 is an efficient solution of (P) with efficient pair (F_1, T_1), and X_2 is another efficient solution with efficient pair (F_2, T_2) such that $F_1 < F_2$ and $T_1 > T_2$ (or $F_1 > F_2$ and $T_1 < T_2$), then every feasible solution $X = \lambda X_1 + (1-\lambda)X_2$, $0 < \lambda < 1$ of (P) is dominated by both X_1 and X_2. Thus an edge with X_1 and X_2 as its two end points cannot be efficient. (An efficient edge being one on which every point is an efficient point for the problem). However, if $(F_1, T_1) = (F_2, T_2) = (F, T)$ (say), then an edge with X_1 and X_2 as its end points is an efficient one, as each point on it gives the pair of values (F, T) of the objective function in (P).

For the problem (P), the following four related bottleneck linear programming problems have been defined:

$$\text{BLP}(F(X)): \quad \underset{X \in S}{\text{Min}}\, F(X)$$

$$\text{BLP}(T(X)): \quad \underset{X \in S}{\text{Min}}\, T(X)$$

$$\text{BLP}(F(X)/T_0): \quad \underset{X \in S}{\text{Min}}\, F(X)$$

where

$$\begin{aligned} f_j &= f_j \quad \text{if} \quad t_j \le T_0 \\ &= \infty \quad \text{if} \quad t_j > T_0, \end{aligned}$$

T_0 being a particular value of $T(X)$.

$$\text{BLP}(T(X)/F_0): \quad \underset{X \in S}{\text{Min}}\, T(X)$$

where

$$\begin{aligned} t_j &= t_j \quad \text{if} \quad f_j \le F_0 \\ &= \infty \quad \text{if} \quad f_j > F_0, \end{aligned}$$

F_0 being a particular value of $F(X)$.

Remark 2: In each of the above four problems, the objective function is a single bottleneck function which is concave and attains its global minimum at an extreme point of S [3, 6, 7, 11, 12].

Remark 3: Let F_1 be the optimal value of $F(X)$ in BLP($F(X)$), and T_1 be the optimal value of $T(X)$ in BLP($T(X)/F_1$). Then it is clear that there does not exist any $X \in S$ with $F(X) < F_1$ or any $X \in S$ with $F(X) = F_1$ and $T(X) < T_1$. Therefore, (F_1, T_1) is an efficient pair for (P). Also, there does not exist an efficient pair (F, T) with $F < F_1$. Similarly, if T_p is the optimal value of $T(X)$ in BLP($T(X)$) and F_p is the optimal value of $F(X)$ in BLP($F(X)/T_p$), then (F_p, T_p) is an efficient pair for (P), and there does not exist an efficient pair (F, T) with $T < T_p$.

(F_1, T_1) and (F_p, T_p) are called the first and last efficient pairs of (P) respectively.

Remark 4: It is clear from the above remark that if S is non-empty, the problem (P), with $F(X)$ and $T(X)$ as conflicting objectives, has at least two efficient solutions.

Henceforth, the kth efficient pair of (P) will be denoted by (F_k, T_k), and the corresponding efficient solution by X_k (or X_k^i, $i = 1, 2, \ldots$, if there are more than one). The efficient pairs are listed in increasing order of values of the first objective function. Let $f_r = \text{Min}\{f_j/f_j > F_k\}$, and define the linear programming problem

$$\text{LP}(f_r, T_k):\quad \underset{X \in S}{\text{Min}}\, Z(X) = \sum_{j=1}^{n} c_j x_j$$

where

$$\begin{aligned} c_j &= 0, f_j < f_r, t_j < T_k \\ &= -1, f_j = f_r, t_j < T_k \\ &= \infty, \text{otherwise.} \end{aligned}$$

Theorem 2: *If the optimal value of $Z(X)$ in* LP(f_r, T_k) *is negative, then f_r is the value of the first objective in the $(k+1)$th efficient pair (F_{k+1}, T_{k+1}), and T_{k+1} is given by the optimal value of $T(X)$ in* BLP$(T(X)/f_r)$.

Proof: Let $\hat{X} = [\hat{x}_j]$ be the optimal solution of LP(f_r, T_k). By hypothesis, $Z(\hat{X}) < 0$. Clearly, $\hat{x}_j = 0\ \forall j : f_j > f_r$ or $t_j \geq T_k$. Also, $\hat{x}_j > 0$ for at least one j with $f_j = f_r$ and $t_j < T_k$. Hence, $F(\hat{X}) = f_r$ and $T(\hat{X}) < T_k$. Now let $X^* = [x_j^*]$ be the optimal solution of BLP$(T(X)/f_r)$, with $T(X^*) = T^*$. Since $\hat{X}$ is also a feasible solution for BLP$(T(X)/f_r)$, therefore $T^* \leq T(\hat{X}) < T_k$. Since $T^* < T_k < \infty$, $x_j^* = 0$ $\forall j : f_j > f_r$. So, $F(X^*) \not> f_r$. Also, $F(X^*) \not< f_r$, for, if $F(X^*) < f_r$, then $(F(X^*), T(X^*))$ dominates (F_k, T_k), which is not possible since (F_k, T_k) is an efficient pair. Thus $F(X^*) = f_r$, and $T(X^*) < T_k$. Hence the $(k+1)$th efficient pair is $(F_{k+1} = f_r, T_{k+1} = T(X^*))$ and $X_{K+1} = X^*$.

Remark 5: If the optimal value of $Z(X)$ in LP(f_r, T_k) is ∞, there does not exist an efficient pair with f_r as value of the first objective function.

Remark 6: The optimal value of $Z(X)$ in LP(f_r, T_k) cannot be zero. If $\hat{X}$ is its optimal solution and $Z(\hat{X}) = 0$, then $F(\hat{X}) \leq F_k$ and $T(\hat{X}) < T_k$ which contradicts the fact that (F_k, T_k) is an efficient pair for (P).

3. ALGORITHM

The algorithm for finding the efficient pairs runs as follows

Step 1. Solve BLP$(F(X))$ and BLP$(T(X))$ [3, 11]. Let F_1 and T_p be the optimal values of $F(X)$ and $T(X)$ respectively in the two problems. Solve BLP$(T(X)/F_1)$ and BLP$(F(X)/T_p)$, and let T_1 and F_p be their respective optimal values. Then (F_1, T_1) and (F_p, T_p) are the first and last efficient pairs respectively. Set as X_1, X_p the optimal solutions of BLP$(T(X)/F_1)$ and BLP$(F(X)/T_p)$ respectively.

Step 2. Arrange the f_j's $(F_1 < f_j < F_p)$ and t_j's $(T_1 > t_j > T_p)$ as:

$$F_1 < f_{j_1} < \cdots < f_{j_r} < F_p,$$

$$T_1 > t_{j_1} > \cdots > t_{j_s} > T_p.$$

Set $k = 2$ and $q = 1$.
Go to Step 3.

Step 3. Solve LP(f_{j_q}, T_{k-1}). If $Z(X) = \infty$, return to Step 3 with next higher value of q. If $Z(X) < 0$, solve BLP$(T(X)/f_{j_q})$. Let X^* be its optimal solution and T^* be the optimal value of $T(X)$. Set $F_k = f_{j_q}$, $T_k = T^*$ and $X_k = X^*$.

Repeat Step 3 with next higher values of k and q. Continue till $f_{j_q} = F_p$ or $T^* = T_p$.

Example: Consider the following problem

$$\text{'Min'}(F(X),\ T(X))$$

subject to

$$x_1 + 2x_2 + x_3 + 2x_4 + 3x_6 = 6$$
$$3x_1 + x_2 + 2x_3 + x_5 + 2x_6 = 5$$
$$x_1 + x_2 + x_3 + x_5 + x_6 = 3,$$
$$x_j \geq 0,\ j = 1, \ldots, 6,$$

where

$$F(X) = \operatorname*{Max}_j f_j(x_j), \qquad f_j(x_j) = f_j,\ x_j > 0$$
$$= 0,\ x_j = 0, \qquad j = 1, \ldots, 6,$$
$$T(X) = \operatorname*{Max}_j t_j(x_j), \qquad t_j(x_j) = t_j,\ x_j > 0$$
$$= 0,\ x_j = 0, \qquad j = 1, \ldots, 6$$

and

$$f_1 = 10, \quad f_2 = 15, \quad f_3 = 28, \quad f_4 = 18, \quad f_5 = 20, \quad f_6 = 23,$$
$$t_1 = 30, \quad t_2 = 15, \quad t_3 = 25, \quad t_4 = 14, \quad t_5 = 27, \quad t_6 = 13.$$

The optimal solution of BLP($F(X)$) is

$$X = (1 \quad 2 \quad 0 \quad 0.5 \quad 0 \quad 0) \quad \text{with} \quad F(X) = 18 = F_1.$$

The optimal solution of BLP($T(X)/F_1$) is

$$X = (1 \quad 2 \quad 0 \quad 0.5 \quad 0 \quad 0) \quad \text{with} \quad T(X^*) = 30.$$

Therefore the first efficient pair is $(18, 30)$ with corresponding efficient solution $X_1 = (1 \quad 2 \quad 0 \quad 0.5 \quad 0 \quad 0)$. The optimal solution of BLP($T(X)$) is

$$X = (0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1) \quad \text{with} \quad T(X) = 25 = T_p.$$

The optimal solution of BLP($F(X)/T_p$) is

$$X = (0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1) \quad \text{with} \quad F(X) = 28 = F_p.$$

Therefore the last efficient pair is $(28, 25)$ with a corresponding efficient solution $X_p = (0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1)$. Now

$$F_1 = 18 < f_{j_1} = 20 < f_{j_2} = 23 < F_p = 28.$$
$$T_1 = 30 > t_{j_1} = 27 > T_p = 25.$$

The optimal value of the objective function in LP($f_{j_1} = 20$, $T_1 = 30$) is ∞. Therefore, there is no efficient pair with $F(X) = 20$. Solve LP($f_{j_2} = 23$, $T_1 = 30$). The optimal value of the objective function in LP$(23, 30)$ is -2. Solve LP($T(X)/23$). The optimal solution is $X^* = (0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2)$ with $T(X^*) = 27$. Thus, $(23, 27)$ is the second efficient pair with a corresponding efficient solution $X_2 = X^*$.

Therefore, three efficient pairs: $(18, 30)$, $(23, 27)$ and $(28, 25)$ have been found.

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